# De Rham Complex for Quantized Irreducible Flag Manifolds

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**Abstract**

It is shown that quantized irreducible flag manifolds possess a canonical $q$-analogue of the de Rham complex. Generalizing the well known situation for the standard Podleś' quantum sphere this analogue is obtained as the universal differential calculus of a distinguished first order differential calculus. The corresponding differential d can be written as a sum of differentials $\partial$ and $\overline{\partial}$. The universal differential calculus corresponding to the first order differential calculi d, $\partial$, and $\overline{\partial}$ are given in terms of generators and relations. Relations to well known quantized exterior algebras are established. The dimensions of the homogeneous components are shown to be the same as in the classical case. The existence of a volume form is proven.

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**Key words.** Quantum groups, quantized flag manifolds, differential calculus.

## 1 Introduction

The theory of quantum groups provides numerous examples of $q$-deformed coordinate rings of spaces with group action. Originally initiated by S. L. Woronowicz there exists by now a rich theory of covariant differential calculi over these comodule algebras.

In A. Connes' more general concept of noncommutative geometry [Con95] spectral triples and in particular the Dirac operator are central notions. One

deals with a representation of an algebra $\mathcal{B}$ on some Hilbert space $\mathcal{H}$ and with an operator $D : \mathcal{H} \to \mathcal{H}$ such that the commutators $\mathrm{d}\, b := [b, D]$, $b \in \mathcal{B}$, lead to a differential graded algebra. It has recently been pointed out, that the theory of quantum groups provides a large class of examples, so called quantized irreducible flag manifolds, which seem to fit well into Connes' framework of noncommutative geometry [LD02],[Krä03], [SW03]. Covariant differential calculi over quantized irreducible flag manifolds have been classified in [HK03b]. There exists a canonical covariant first order differential calculus on these spaces, which turned out to correspond to the Dirac operator constructed in [Krä03]. It is therefore natural to investigate the corresponding higher order differential calculi.

Higher order differential calculi have previously been studied from several points of view. In [SV98], [SSV99] L. L. Vaksman and his coworkers presented a canonical construction of differential forms on quantum prehomogeneous vector spaces. As these spaces are big cells of the flag manifolds under consideration here, these differential calculi are closely related to those investigated in the present paper.

An approach to noncommutative geometry modelled on classical geometry and compatible with the intrinsic structure of quantum groups and quantum spaces has been put forward by T. Brzeziński and S. Majid in [BM93], [BM00]. In this approach the notion of differential calculus is one starting point. A similar point of view has been adopted by M. Đurđević in [Đu96], [Đu97].

In the early days of quantum groups there appeared several examples of differential calculi which in many respects behave as the de Rham complex over the corresponding commutative algebras [Wor87], [PW89], [WZ91]. Yet it became clear that imposing covariance, i.e. compatibility with a quantum group action, one can not expect that such a calculus exists for an arbitrary quantum space. Nevertheless, there soon existed a well developed theory of covariant differential calculi on quantum groups while for quantum spaces similar results have only recently been established [Her02], [HK03a]. Apart from the various examples of differential calculi on quantum groups (cp. references in [KS97]) and quantum vector spaces as above, only differential calculi over Podleś' quantum spheres [Pod92] and Vaksman-Soibelman-spheres [Wel98] have been in detail investigated.

In the present paper differential calculi over quantized irreducible flag manifolds are studied in detail. Contrary to the situation for quantum groups and quantum vector spaces for this large class of examples the modules of differential forms are generally not free over the coordinate algebra. Note, however, that the general theory implies that being covariant these modules are projective. The differential calculus constructed here is a close analogue

of the de Rham complex over the corresponding complex manifold. As in complex geometry the differential $\mathrm{d}$ can be decomposed into the sum of differentials $\partial$ and $\overline{\partial}$. The universal differential calculus corresponding to the first order differential calculi $\mathrm{d}$, $\partial$, and $\overline{\partial}$ are given in terms of generators and relations. The dimensions of the homogeneous components are shown to be the same as in the classical case. In particular the differential $\mathrm{d}$ admits a uniquely determined volume form of degree $2M$, where $M$ is the complex dimension of the manifold. The fibers of the differential calculi over the classical point $\varepsilon$ of the quantized flag manifold are shown to be isomorphic to well known examples of quantized exterior algebras [FRT89].

The organization of the paper is as follows. In Chapter 2 we mainly recall the relevant notions from the theory of quantum groups, quantum homogeneous spaces, and differential calculus. It is explained in Section 2.3.5 how the notion of quantum tangent space introduced in [HK03a] can be employed to determine the homogeneous component of degree two of the universal differential calculus corresponding to a given finite dimensional covariant first order differential calculus.

Chapter 3 is devoted to the construction and investigation of the desired differential calculus on quantized flag manifolds. In Section 3.1 the various quantized coordinate rings associated to flag manifolds are recalled. On the one hand there exist homogeneous coordinate rings $S_q[G/P_S]$ and $S_q[G/P_S^{\mathrm{op}}]$. On the other hand the quantized algebra of functions $\mathbb{C}_q[G/L_S]$ on the quotient $G/L_S$ of the Lie group $G$ by the Levi factor $L_S$ of the parabolic subgroup $P_S \subset G$ is considered. It is crucial, as observed in [Sto02], [HK03b], that certain products of generators of $S_q[G/P_S]$ and $S_q[G/P_S^{\mathrm{op}}]$ generate $\mathbb{C}_q[G/L_S]$. This observation allows the construction of first order differential calculi $\Gamma_\partial$, $\Gamma_{\overline{\partial}}$, and $\Gamma_\mathrm{d}$ over $\mathbb{C}_q[G/L_S]$ via the construction of first order differential calculi over $S_q[G/P_S]$ and $S_q[G/P_S^{\mathrm{op}}]$ in Section 3.2. All first order differential calculi over $\mathbb{C}_q[G/L_S]$ constructed in this section are also given in terms of generators and relations, their quantum tangent spaces are determined, and their dimensions are calculated.

Finally, Section 3.3 is devoted to the corresponding universal differential calculi $\Gamma^\wedge_{\partial,\mathrm{u}}$, $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$, and $\Gamma^\wedge_{\mathrm{d},\mathrm{u}}$. Again, first the situation for $\Gamma_\partial$ and $\Gamma_{\overline{\partial}}$ is analyzed in detail. Then it is shown that the differentials $\partial$ and $\overline{\partial}$ can be extended to the universal differential calculus $\Gamma^\wedge_{\mathrm{d},\mathrm{u}}$. Thus one can reduce statements about $\Gamma^\wedge_{\mathrm{d},\mathrm{u}}$ to the corresponding statements about the the universal differential calculi $\Gamma^\wedge_{\partial,\mathrm{u}}$ and $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$.

All algebras considered in this paper are unital $\mathbb{C}$-algebras, likewise all vector spaces are defined over $\mathbb{C}$.

Throughout this paper several filtrations are defined in the following way.

Let $A$ denote an algebra generated by the elements of a set $Z$ and $\mathcal{S}$ a totally ordered abelian semigroup. Then any map $\deg : Z \to \mathcal{S}$ defines a filtration $\mathcal{F}$ of the algebra $A$ as follows. An element $a \in A$ belongs to $\mathcal{F}_n$, $n \in \mathcal{S}$, if and only if it can be written as a polynomial in the elements of $Z$ such that every occurring summand $a_{1\dots k} z_1 \dots z_k$, $a_{1\dots k} \in \mathbb{C}$, $z_i \in Z$, satisfies $\sum_{j=1}^k \deg(z_j) \le n$. Instead of $a \in \mathcal{F}_n$ by slight abuse of notation we will also write $\deg(a) = n$.

For any Hopf algebra $H$ the symbols $\varDelta$, $\varepsilon$, and $\kappa$ will denote the coproduct, counit, and antipode, respectively. Sweedler notation for coproducts $\varDelta a = a_{(1)} \otimes a_{(2)}$, $a \in H$, will be used. If the antipode $\kappa$ is invertible we will frequently identify left and right $H$-module structures on a vector space $V$ by $vh = \kappa^{-1}(h)v$, $v \in V$, $h \in H$. The symbol $H^{\mathrm{op}}$ will denote the corresponding Hopf algebra with opposite multiplication.

# 2 Preliminaries

## 2.1 Notations

First, to fix notations some general notions related to Lie algebras are recalled. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a fixed Cartan subalgebra. Let $R \subset \mathfrak{h}^*$ denote the root system associated with $(\mathfrak{g}, \mathfrak{h})$. Choose an ordered basis $\pi = \{\alpha_1, \dots, \alpha_r\}$ of simple roots for $R$ and let $R^+$ (resp. $R^-$) be the set of positive (resp. negative) roots with respect to $\pi$. Moreover, let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the corresponding triangular decomposition. Identify $\mathfrak{h}$ with its dual via the Killing form. The induced nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ is denoted by $(\cdot, \cdot)$. The root lattice $Q = \mathbb{Z}R$ is contained in the weight lattice $P = \{\lambda \in \mathfrak{h}^* \,|\, (\lambda, \alpha_i)/d_i \in \mathbb{Z} \,\forall \alpha_i \in \pi\}$ where $d_i := (\alpha_i, \alpha_i)/2$. In order to avoid roots of the deformation parameter $q$ in the following sections we rescale $(\cdot, \cdot)$ such that $(\cdot, \cdot) : P \times P \to \mathbb{Z}$.

For $\mu, \nu \in P$ we will write $\mu \succ \nu$ if $\mu - \nu$ is a sum of positive roots and $\mu \succnsim \nu$ if $\mu \succ \nu$ and $\mu \neq \nu$. As usual we define $Q^+ := \{\mu \in Q \,|\, \mu \succ 0\}$. The height $\mathrm{ht} : Q^+ \to \mathbb{N}_0$ is given by $\mathrm{ht}(\sum_{i=1}^r n_i \alpha_i) = \sum_{i=1}^r n_i$.

The fundamental weights $\omega_i \in \mathfrak{h}^*$, $i = 1, \dots, r$ are characterized by $(\omega_i, \alpha_j)/d_j = \delta_{ij}$. Let $P^+$ denote the set of dominant weights, i. e. the $\mathbb{N}_0$-span of $\{\omega_i \,|\, i = 1, \dots, r\}$. Recall that $(a_{ij}) := (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathfrak{g}$ with respect to $\pi$.

For $\mu \in P^+$ let $V(\mu)$ denote the uniquely determined finite dimensional irreducible left $\mathfrak{g}$-module with highest weight $\mu$. More explicitly there exists a nontrivial vector $v_\mu \in V(\mu)$ satisfying

$$Ev_\mu = 0, \quad Hv_\mu = \mu(H)v_\mu \qquad \text{for all } H \in \mathfrak{h},\ E \in \mathfrak{n}_+. \tag{1}$$

For any weight vector $v \in V(\mu)$ let $\mathrm{wt}(v) \in P$ denote the weight of $v$, i.e. $Hv = \mathrm{wt}(v)(H)v$. In particular $\mathrm{wt}(v_1) - \mathrm{wt}(v_2) \in Q$ for all weight vectors $v_1, v_2 \in V(\mu)$.

Let $G$ denote the connected simply connected complex Lie group with Lie algebra $\mathfrak{g}$. For any set $S \subset \pi$ of simple roots define $R_S^\pm := \mathbb{Z}S \cap R^\pm$ and $\overline{R_S^\pm} := R^\pm \setminus R_S^\pm$. Let $P_S$ and $P_S^{\mathrm{op}}$ denote the corresponding standard parabolic subgroups of $G$ with Lie algebra

$$\mathfrak{p}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \cup R_S^-} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_S^{\mathrm{op}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^- \cup R_S^+} \mathfrak{g}_\alpha. \tag{2}$$

Moreover,

$$\mathfrak{l}_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in R_S^+ \cup R_S^-} \mathfrak{g}_\alpha$$

is the Levi factor of $\mathfrak{p}_S$ and $L_S = P_S \cap P_S^{\mathrm{op}} \subset G$ denotes the corresponding subgroup. Later on by slight abuse of notation we will also write $i \in S$ instead of $\alpha_i \in S$.

The generalized flag manifold $G/P_S$ is called irreducible if the adjoint representation of $\mathfrak{p}_S$ on $\mathfrak{g}/\mathfrak{p}_S$ is irreducible. Equivalently, $S = \pi \setminus \{\alpha_i\}$ where $\alpha_i$ appears in any positive root with coefficient at most one. For a complete list of all irreducible flag manifolds consult e.g. [BE89, p. 27].

## 2.2 Quantum Groups

### 2.2.1 Definition of $U_q(\mathfrak{g})$

We keep the notations of the previous section. Let $q \in \mathbb{C} \setminus \{0\}$ be not a root of unity. The $q$-deformed universal enveloping algebra $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$ can be defined to be the complex algebra with generators $K_i, K_i^{-1}, E_i, F_i$, $i = 1, \ldots, r$, and relations

$$\begin{gathered}
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \\
K_i E_j = q^{(\alpha_i, \alpha_j)} E_j K_i, \quad K_i F_j = q^{-(\alpha_i, \alpha_j)} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \quad i \neq j, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \quad i \neq j,
\end{gathered} \tag{3}$$

where $q_i := q^{d_i}$ and the $q$-deformed binomial coefficients are defined by

$$\binom{n}{k}_q = \frac{[n]_q[n-1]_q \dots [n-k+1]_q}{[1]_q[2]_q \dots [k]_q}, \qquad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

The algebra $U_q(\mathfrak{g})$ obtains a Hopf algebra structure by

$$\begin{array}{lll} \Delta K_i = K_i \otimes K_i, & \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, & \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ \epsilon(K_i) = 1, & \epsilon(E_i) = 0, & \epsilon(F_i) = 0, \\ \kappa(K_i) = K_i^{-1}, & \kappa(E_i) = -E_i K_i^{-1}, & \kappa(F_i) = -K_i F_i. \end{array} \tag{4}$$

Let $U_q(\mathfrak{n}_+), U_q(\mathfrak{b}_+), U_q(\mathfrak{n}_-), U_q(\mathfrak{b}_-) \subset U_q(\mathfrak{g})$ denote the subalgebras generated by $\{E_i \,|\, i = 1, \dots, r\}$, $\{E_i, K_i, K_i^{-1} \,|\, i = 1, \dots, r\}$, $\{F_i \,|\, i = 1, \dots, r\}$, and $\{F_i, K_i, K_i^{-1} \,|\, i = 1, \dots, r\}$, respectively. Moreover, for $\beta \in Q^+$ we will write $U_q^\beta(\mathfrak{n}_+) := \{x \in U_q(\mathfrak{n}_+) \,|\, K_i x K_i^{-1} = q^{(\beta,\alpha_i)} x\}$ and $U_q^\beta(\mathfrak{n}_-) := \{x \in U_q(\mathfrak{n}_-) \,|\, K_i x K_i^{-1} = q^{-(\beta,\alpha_i)} x\}$.

### 2.2.2 Type 1 Representations

For $\mu \in P^+$ let $V(\mu)$ denote the uniquely determined finite dimensional irreducible left $U_q(\mathfrak{g})$-module with highest weight $\mu$. More explicitly, there exists a highest weight vector $v_\mu \in V(\mu) \setminus \{0\}$ satisfying

$$E_i v_\mu = 0, \quad K_i v_\mu = q^{(\mu,\alpha_i)} v_\mu \qquad \text{for all } i = 1, \dots, r. \tag{5}$$

A finite dimensional $U_q(\mathfrak{g})$-module $V$ is called *of type 1* if $V \cong \bigoplus_i V(\mu_i)$ is isomorphic to a direct sum of finitely many $V(\mu_i)$, $\mu_i \in P^+$. The category $\mathcal{C}$ of $U_q(\mathfrak{g})$-modules of type 1 is a tensor category. By this we mean that $\mathcal{C}$ contains the trivial $U_q(\mathfrak{g})$-module $V(0)$ and satisfies

$$X, Y \in \mathcal{C} \Rightarrow X \oplus Y,\ X \otimes Y,\ X^* \in \mathcal{C} \tag{6}$$

where $(uf)(x) := f(\kappa(u)x)$ for all $u \in U$, $f \in X^*$, $x \in X$.

During subsequent considerations we will meet various natural right $U_q(\mathfrak{g})$-modules. As indicated at the end of the introduction we will always endow a right $U_q(\mathfrak{g})$-module $V$ with the left $U_q(\mathfrak{g})$-action defined by

$$uv := v\kappa(u) \qquad \forall u \in U, v \in V.$$

### 2.2.3 The Braiding

The category $\mathcal{C}$ in Section 2.2.2 is a braided category. Unfortunately the relevant section in our main reference [KS97, 8.3.3] lacks notational consistency. To be able to derive additional properties of the braiding we recall its construction in some detail.

Recall that the dual pairing $\langle\cdot,\cdot\rangle : U_q(\mathfrak{b}_+)\times U_q(\mathfrak{b}_-)^{\mathrm{op}} \to \mathbb{C}$ of Hopf algebras [KS97, 6.3.1] remains non-degenerate when restricted to $U_q(\mathfrak{n}_+) \times U_q(\mathfrak{n}_-)^{\mathrm{op}}$ and satisfies $\langle a, b\rangle = 0$ for all $a \in U_q^{\mu}(\mathfrak{n}_+)$, $b \in U_q^{-\nu}(\mathfrak{n}_-)$, $\mu \neq \nu$. Let $C_\beta \in U_q^{\beta}(\mathfrak{n}_+) \otimes U_q^{-\beta}(\mathfrak{n}_-)$ denote the canonical element with respect to $\langle\cdot,\cdot\rangle$, i.e. $C_\beta = \sum_i a_i \otimes b_i$ where $\{a_i\}$ is a basis of $U_q^{\beta}(\mathfrak{n}_+)$ and $\langle a_i, b_j\rangle = \delta_{ij}$. Define

$$\mathfrak{R} := \sum_{\beta\in Q^+} (K_\beta \otimes 1)(\kappa^{-1} \otimes 1)C_\beta \qquad \in \bar{U}_q^+(\mathfrak{g})\bar{\otimes}\bar{U}_q^+(\mathfrak{g}), \tag{7}$$
$$\mathfrak{R}^{-1} := \sum_{\beta\in Q^+} C_\beta \qquad \in \bar{U}_q^+(\mathfrak{g})\bar{\otimes}\bar{U}_q^+(\mathfrak{g})$$

where $\bar{U}_q^+(\mathfrak{g})$ and $\bar{U}_q^+(\mathfrak{g})\bar{\otimes}\bar{U}_q^+(\mathfrak{g})$ are defined in [KS97, 6.3.3]. There exists an automorphism $\Phi$ of the algebra $\bar{U}_q^+(\mathfrak{g})\bar{\otimes}\bar{U}_q^+(\mathfrak{g})$ such that

$$\begin{aligned} \Phi(K_i \otimes 1) &= K_i \otimes 1, & \Phi(1 \otimes K_i) &= 1 \otimes K_i,\\ \Phi(E_i \otimes 1) &= E_i \otimes K_i^{-1}, & \Phi(1 \otimes E_i) &= K_i^{-1} \otimes E_i,\\ \Phi(F_i \otimes 1) &= F_i \otimes K_i, & \Phi(1 \otimes F_i) &= K_i \otimes F_i. \end{aligned}$$

One verifies that $\mathfrak{R}$ and $\Phi$ satisfy the properties stated in [KS97, Theorem 8.18]. We suggest first to check the corresponding properties of $\mathfrak{R}^{-1}$.

For all $V, W \in \mathcal{C}$ the action of $\mathfrak{R}$ on $V \otimes W$ induces a $U_q(\mathfrak{g})$-module isomorphism

$$\hat{R}_{V,W} : V \otimes W \to W \otimes V, \qquad \hat{R}_{V,W} := \tau \circ B_{V,W}(\mathfrak{R}(v \otimes w)) \tag{8}$$

where $\tau$ denotes the twist $\tau(v \otimes w) = w \otimes v$, and $B_{V,W}(v \otimes w) = q^{(\mu,\nu)} v \otimes w$ for weight vectors $v \in V$ and $w \in W$ of weight $\mu$ and $\nu$, respectively. The family $(\hat{R}_{VW})$ defines a braiding in $\mathcal{C}$. To simplify notation we will also write $\hat{R}_{\mu,\nu} := \hat{R}_{V(\mu),V(\nu)}$ if $\mu, \nu \in P^+$.

**Remark 2.1.** An explicit formula for $\mathfrak{R}$ is given in [KS97, 8.3.3, above Thm. 18]. Note that the braiding $\hat{R}$ is uniquely determined if one demands that (8) is a $U_q(\mathfrak{g})$-module homomorphism satisfying

$$\hat{R}_{V,W}(v \otimes w) = q^{(\mathrm{wt}(v),\mathrm{wt}(w))} w \otimes v + \sum_i w_i \otimes v_i \tag{9}$$

where $\mathrm{wt}(w) \succ \mathrm{wt}(w_i)$ and $\mathrm{wt}(v_i) \succ \mathrm{wt}(v)$. Indeed, if $v_{\max} \in V$ is a highest weight vector then $\hat{R}_{V,W}(v_{\max} \otimes w)$ is uniquely determined by (9) for any $w \in W$. The property of being a $U_q(\mathfrak{g})$-module homomorphism then fixes $\hat{R}_{V,W}$ on all of $V \otimes W$.

For this reason (7) should coincide with the explicit expression in [KS97, 8.3.3]. It is straightforward to check that coefficients of the terms $E_i \otimes F_i$ of the two expressions are identical. Coefficients of higher order terms of the explicit expression of $\mathfrak{R}$ will not be used in this paper.

### 2.2.4 Restriction of the Braiding to $U_q(\mathfrak{l}_S)$

Let $S \subset \pi$ and let $\mathfrak{k}_S := [\mathfrak{l}_S, \mathfrak{l}_S] \subset \mathfrak{g}$ denote the semisimple part of $\mathfrak{l}_S \subset \mathfrak{p}_S \subset \mathfrak{g}$. Define $U_q(\mathfrak{k}_S)$ and $U_q(\mathfrak{l}_S)$ to be the Hopf subalgebras of $U_q(\mathfrak{g})$ generated by the sets $\{K_j, K_j^{-1}, E_j, F_j \,|\, j \in S\}$ and $\{K_i, K_i^{-1}, E_j, F_j \,|\, j \in S,\, i = 1, \dots, r\}$, respectively.

As above the tensor category $\mathcal{C}^{\mathfrak{k}}$ of type 1 representations of $U_q(\mathfrak{k}_S)$ is braided with braiding $\hat{R}^{\mathfrak{k}}_{V,W}$. Moreover, let $(\cdot,\cdot)_{\mathfrak{k}}$ denote the uniquely determined bilinear form on the weight lattice corresponding to $\mathfrak{k}_S$ such that $(\alpha,\beta)_{\mathfrak{k}} = (\alpha,\beta)$ for all simple roots $\alpha,\beta \in S$.

The following Lemma will be used only in the proof of Proposition 3.11 at the very end of this paper.

**Lemma 2.2.** *Let $V = V(\nu)$ and $W = V(\mu)$ be irreducible $U_q(\mathfrak{g})$-modules and let $V' = V(\nu') \subset V$ and $W' = V(\mu') \subset W$ be irreducible $U_q(\mathfrak{l}_S)$-submodules. Let $p_V : V \to V'$ and $p_W : W \to W'$ denote surjective $U_q(\mathfrak{l}_S)$-module homomorphisms satisfying $p_V^2 = p_V$ and $p_W^2 = p_W$, respectively. Then*

$$(p_W \otimes p_V)\hat{R}_{V,W}|_{V'\otimes W'} = q^{(\nu-\gamma_1,\mu-\gamma_2)-(\nu',\mu')_{\mathfrak{k}}}\hat{R}^{\mathfrak{k}}_{V',W'}$$

*where the highest weight vectors of $V'$ and $W'$ have weight $\nu - \gamma_1$ in $V$ and $\mu - \gamma_2$ in $W$, $\gamma_1, \gamma_2 \in Q^+$, respectively.*

*Proof.* In analogy to (7) one has an element

$$\mathfrak{R}^{\mathfrak{k}} := \sum_{\beta \in \mathbb{Z}S \cap Q^+} (K_\beta \otimes 1)(\kappa^{-1} \otimes 1)C_\beta \in \bar{U}_q^+(\mathfrak{k}_S)\bar{\otimes}\bar{U}_q^+(\mathfrak{k}_S)$$

and linear maps

$$B^{\mathfrak{k}}_{V',W'} : V' \otimes W' \to V' \otimes W', \quad v \otimes w \mapsto q^{(\mu,\nu)_{\mathfrak{k}}} v \otimes w$$

where $V', W' \in \mathcal{C}^{\mathfrak{k}}$ and $v$ and $w$ are weight vectors of weight $\mu$ and $\nu$, respectively. Then for all $v' \in V'$, $w' \in W'$ one obtains

$$\begin{aligned}
(p_W \otimes p_V)\hat{R}_{V,W}(v' \otimes w') &\overset{(8)}{=} \tau \circ B_{V,W}(p_V \otimes p_W)(\mathfrak{R}(v' \otimes w')) \\
&\overset{(7)}{=} \tau \circ B_{V,W}(\mathfrak{R}^{\mathfrak{k}}(v' \otimes w')) \\
&= B_{W,V} \circ (B^{\mathfrak{k}}_{W',V'})^{-1} \circ \tau \circ B^{\mathfrak{k}}_{V',W'}(\mathfrak{R}^{\mathfrak{k}}(v' \otimes w')) \\
&= B_{W,V} \circ (B^{\mathfrak{k}}_{W',V'})^{-1} \circ \hat{R}^{\mathfrak{k}}_{V',W'}(v' \otimes w').
\end{aligned}$$

Note that by definition of $\gamma_1$ and $\gamma_2$ one has $(\nu-\gamma_1,\alpha_i) = (\nu',\alpha_i)_\mathfrak{k}$ and $(\mu-\gamma_1,\alpha_i) = (\mu',\alpha_i)_\mathfrak{k}$ for all $\alpha_i \in S$. Now the claim of the lemma follows from

$$\begin{aligned}(\nu-\gamma_1-\beta_1,&\mu-\gamma_2-\beta_2) - (\nu'-\beta_1,\mu'-\beta_2)_\mathfrak{k}\\ &=(\nu-\gamma_1,\mu-\gamma_2) - (\beta_1,\mu-\gamma_2) - (\nu-\gamma_1,\beta_2) + (\beta_1,\beta_2)\\ &\quad - (\nu',\mu')_\mathfrak{k} + (\beta_1,\mu')_\mathfrak{k} + (\nu',\beta_2)_\mathfrak{k} - (\beta_1,\beta_2)_\mathfrak{k}\\ &=(\nu-\gamma_1,\mu-\gamma_2) - (\nu',\mu')_\mathfrak{k}\end{aligned}$$

for all $\beta_1,\beta_2 \in \mathbb{Z}S \cap Q^+$. $\square$

### 2.2.5 $R$-Matrices

To write coordinate algebras of quantized flag manifolds in terms of generators and relations it will be helpful to introduce additional notations for certain special cases of $\hat{R}$. For $\lambda = \sum_{i\notin S}\omega_i$ set $N := \dim V(\lambda)$ and abbreviate $I := \{1,\dots,N\}$. Choose a basis $\{v_i \,|\, i \in I\}$ of weight vectors of $V(\lambda)$ and let $\{f_i \,|\, i \in I\}$ be the corresponding dual basis. Define matrices $\hat{R}$, $\check{R}$, $\acute{R}^-$ and $\grave{R}^-$ by

$$\hat{R}_{\lambda,\lambda}(v_i\otimes v_j) =: \sum_{k,l\in I}\hat{R}^{kl}_{ij}v_k\otimes v_l, \qquad \hat{R}_{-w_0\lambda,-w_0\lambda}(f_i\otimes f_j) =: \sum_{k,l\in I}\check{R}^{kl}_{ij}f_k\otimes f_l,$$

$$\hat{R}_{-w_0\lambda,\lambda}(f_i\otimes v_j) =: \sum_{k,l\in I}\acute{R}^{-kl}_{\ \ ij}v_k\otimes f_l, \qquad \hat{R}_{\lambda,-w_0\lambda}(v_i\otimes f_j) =: \sum_{k,l\in I}\grave{R}^{-kl}_{\ \ ij}f_k\otimes v_l.$$

Alternatively

$$(f_i\otimes f_j)\circ\hat{R}_{\lambda,\lambda} = \sum_{k,l\in I}\hat{R}^{ij}_{kl}f_k\otimes f_l, \qquad (v_i\otimes v_j)\circ\hat{R}_{-w_0\lambda,-w_0\lambda} = \sum_{k,l\in I}\check{R}^{ij}_{kl}v_k\otimes v_l,$$

$$(f_i\otimes v_j)\circ\hat{R}_{-w_0\lambda,\lambda} = \sum_{k,l\in I}\acute{R}^{-ij}_{\ \ kl}v_k\otimes f_l, \qquad (v_i\otimes f_j)\circ\hat{R}_{\lambda,-w_0\lambda} = \sum_{k,l\in I}\grave{R}^{-ij}_{\ \ kl}f_k\otimes v_l,$$

where the elements of $V(\lambda)$ are considered as functionals on $V(\lambda)^*$. Let $\hat{R}^-$, $\check{R}^-$, $\grave{R}$ and $\acute{R}$ denote the inverse of the matrix $\hat{R}$, $\check{R}$, $\acute{R}^-$ and $\grave{R}^-$, respectively.

By (7) the matrix $\hat{R}$ has the property that $\hat{R}^{ij}_{kl} \neq 0$ implies that $i = l, j = k$ or both $\mathrm{wt}(v_j) \gnapprox \mathrm{wt}(v_k)$ and $\mathrm{wt}(v_l) \gnapprox \mathrm{wt}(v_i)$. Therefore we associate to $\hat{R}$ the symbol $<$ which denotes the positions of the larger weights. Similar properties are fulfilled for the other types of $R$-matrices. For example, the relation $\acute{R}^{-ij}_{\ \ kl} \neq 0$ implies that $i = l, j = k$ or both $\mathrm{wt}(v_k) \gnapprox \mathrm{wt}(v_j)$ and $\mathrm{wt}(v_l) \gnapprox \mathrm{wt}(v_i)$. We collect these properties in the following table.

| $\hat{R}$ | $\hat{R}^{-}$ | $\acute{R}$ | $\acute{R}^{-}$ | $\check{R}$ | $\check{R}^{-}$ | $\grave{R}$ | $\grave{R}^{-}$ |
|---|---|---|---|---|---|---|---|
| $<$ | $>$ | $\vee$ | $\wedge$ | $>$ | $<$ | $\wedge$ | $\vee$ |

(10)

### 2.2.6 The $q$-Deformed Coordinate Ring $\mathbb{C}_q[G]$

The $q$-deformed coordinate ring $\mathbb{C}_q[G]$ is defined to be the subspace of the linear dual $U_q(\mathfrak{g})^*$ spanned by the matrix coefficients of the finite dimensional irreducible representations $V(\mu)$, $\mu \in P^+$. For $v \in V(\mu)$, $f \in V(\mu)^*$ the matrix coefficient $c^\mu_{f,v} \in U_q(\mathfrak{g})^*$ is defined by

$$c^\mu_{f,v}(X) = f(Xv).$$

The linear span of matrix coefficients of $V(\mu)$

$$C^{V(\mu)} = \mathrm{Lin}_{\mathbb{C}}\{c^\mu_{f,v} \,|\, v \in V(\mu), f \in V(\mu)^*\} \tag{11}$$

obtains a $U_q(\mathfrak{g})$-bimodule structure by

$$(Yc^\mu_{f,v}Z)(X) = f(ZXYv) = c^\mu_{fZ,Yv}(X). \tag{12}$$

Here $V(\mu)^*$ is considered as a right $U_q(\mathfrak{g})$-module. Note that by construction

$$\mathbb{C}_q[G] \cong \bigoplus_{\mu \in P^+} C^{V(\mu)} \tag{13}$$

is a Hopf algebra and the pairing

$$\mathbb{C}_q[G] \otimes U_q(\mathfrak{g}) \to \mathbb{C} \tag{14}$$

is nondegenerate.

### 2.2.7 Quantum Homogeneous Spaces

We recall the class of quantum homogeneous spaces considered in [MS99]. Let $U$ denote a Hopf algebra over $\mathbb{C}$ with bijective antipode $\kappa$ and $K \subset U$ a right coideal subalgebra with right coaction $\Delta_K : K \to K \otimes U$. Consider a tensor category $\mathcal{C}$ of finite dimensional left $U$-modules. By this we mean that $\mathcal{C}$ is a class of finite dimensional left $U$-modules containing the trivial $U$-module via $\varepsilon$ and satisfying (6).

Let $\mathcal{A} := U^\circ_{\mathcal{C}}$ denote the dual Hopf algebra generated by the matrix coefficients of all $U$-modules in $\mathcal{C}$. Assume that $\mathcal{A}$ separates the elements of

$U$. Assume further that the antipode of $\mathcal{A}$ is bijective. Note that this is equivalent to

$$X \in \mathcal{C} \Rightarrow {}^*X \in \mathcal{C}$$

where ${}^*X = X^*$ as a vector space and $(uf)(x) := f(\kappa^{-1}(u)x)$ for all $u \in U$, $f \in {}^*X$, $x \in X$.

Define a left coideal subalgebra $\mathcal{B} \subset \mathcal{A}$ by

$$\mathcal{B} := \{b \in \mathcal{A} \,|\, b_{(1)}\, b_{(2)}(k) = \varepsilon(k)b \quad \text{for all } k \in K\}. \tag{15}$$

Assume $K$ to be $\mathcal{C}$-semisimple, i.e. the restriction of any $U$-module in $\mathcal{C}$ to the subalgebra $K \subset U$ is isomorphic to the direct sum of irreducible $K$-modules. In full analogy to [MS99, Thm. 2.2 (2)] this implies that $\mathcal{A}$ is a faithfully flat $\mathcal{B}$-module.

#### 2.2.8 Categorial Equivalence

Assume $\mathcal{B} \hookrightarrow \mathcal{A}$ to be a left coideal subalgebra of a Hopf algebra $\mathcal{A}$ with bijective antipode and define $\mathcal{B}^+ := \{b \in \mathcal{B} \,|\, \varepsilon(b) = 0\}$. Then $\overleftarrow{\mathcal{A}} := \mathcal{A}/\mathcal{B}^+\mathcal{A}$ and $\overrightarrow{\mathcal{A}} := \mathcal{A}/\mathcal{A}\mathcal{B}^+$ are right and left $\mathcal{A}$-module coalgebras, respectively. Let ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ and ${}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$ denote the category of left $\mathcal{A}$-covariant left $\mathcal{B}$-modules and of left $\overleftarrow{\mathcal{A}}$-comodules, respectively. Recall that for any coalgebra $C$ the cotensor product of a left $C$-comodule $P$ and a right $C$-comodule $Q$ is defined by

$$P\square_C Q := \left\{ \sum_i p_i \otimes q_i \in P \otimes Q \,\middle|\, \sum_i p_{i(0)} \otimes p_{i(1)} \otimes q_i = \sum_i p_i \otimes q_{i(-1)} \otimes q_{i(0)} \right\}.$$

There exist functors

$$\Phi : {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M} \to {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}, \qquad \Phi(\Gamma) = \Gamma/\mathcal{B}^+\Gamma,$$
$$\Psi : {}^{\overleftarrow{\mathcal{A}}}\mathcal{M} \to {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}, \qquad \Psi(V) = \mathcal{A}\square_{\overleftarrow{\mathcal{A}}} V.$$

Here for any $\Gamma \in {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ the left $\overleftarrow{\mathcal{A}}$-comodule structure on $\Gamma/\mathcal{B}^+\Gamma$ is induced by the left $\mathcal{A}$-comodule structure of $\Gamma$. Moreover, the left $\mathcal{B}$-module and the left $\mathcal{A}$-comodule structures of $\mathcal{A}\square_{\overleftarrow{\mathcal{A}}} V$ are defined on the first tensor factor.

**Theorem 2.3.** [Tak79, Theorem 1] *With the notions as above suppose that $\mathcal{A}$ is a faithfully flat right $\mathcal{B}$-module. Then $\Phi$ and $\Psi$ are mutually inverse equivalences of categories.*

**Remark 2.4.** (i) The functor $\Phi$ is equivalent to $\Phi' : {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M} \to {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$ defined by

$$\begin{aligned}\Phi'(\Gamma) &:= {}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma)\\ &:= \left\{ \sum_i a_i \otimes \rho_i \,\middle|\, \sum_i a_{i(1)}\rho_{i(-1)} \otimes a_{i(2)} \otimes \rho_{i(0)} = 1 \otimes \sum_i a_i \otimes \rho_i \right\}\end{aligned}$$

where the left $\overleftarrow{\mathcal{A}}$-comodule structure on ${}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma)$ is given by

$$\Delta_L\left(\sum_i a_i \otimes \rho_i\right) = \sum_i \kappa^{-1}(a_{i(2)}) \otimes (a_{i(1)} \otimes \rho_i).$$

The isomorphism ${}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma) \to \Gamma/\mathcal{B}^+\Gamma$ is defined by $\sum_i a_i \otimes \rho_i \mapsto \sum_i \varepsilon(a_i)\overline{\rho_i}$. To verify injectivity note that ${}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma) = \{\kappa(\rho_{(-1)}) \otimes \rho_{(0)} \mid \rho \in \Gamma\}$.

(ii) The functor $\Psi$ is equivalent to $\Psi' : {}^{\overleftarrow{\mathcal{A}}}\mathcal{M} \to {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ defined by

$$\begin{aligned}\Psi'(V) &:= (\mathcal{A} \otimes V)^{\mathrm{co}\overrightarrow{\mathcal{A}}}\\ &:= \left\{ \sum_i a_i \otimes v_i \in \mathcal{A} \otimes V \,\middle|\, \sum_i a_{i(1)} \otimes v_{i(0)} \otimes a_{i(2)}\kappa(v_{i(-1)}) = \sum_i a_i \otimes v_i \otimes 1 \right\}.\end{aligned}$$

Here the left $\mathcal{B}$-module and left $\mathcal{A}$-comodule structure on $\Psi'(V)$ is given by

$$b\left(\sum_i a_i \otimes v_i\right) = \sum_i (ba_i) \otimes v_i, \quad \Delta_L\left(\sum_i a_i \otimes v_i\right) = \sum_i a_{i(1)} \otimes (a_{i(2)} \otimes v_i)$$

for all $b \in \mathcal{B}$ and $\sum_i a_i \otimes v_i \in (\mathcal{A} \otimes V)^{\mathrm{co}\overrightarrow{\mathcal{A}}}$.

(iii) In the situation of Theorem 2.3 the coalgebra $\overleftarrow{\mathcal{A}}$ is cosemisimple. Therefore any $\Gamma \in {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ is a projective left $\mathcal{B}$-module.

## 2.3 Differential Calculus

### 2.3.1 First Order Differential Calculus

For the convenience of the reader the notion of differential calculus from [Wor89] is recalled. A *first order differential calculus* (FODC) over an algebra $\mathcal{B}$ is a $\mathcal{B}$-bimodule $\Gamma$ together with a $\mathbb{C}$-linear map

$$\mathrm{d} : \mathcal{B} \to \Gamma$$

such that $\Gamma = \mathrm{Lin}_{\mathbb{C}}\{a\,\mathrm{d}b\,c \mid a, b, c \in \mathcal{B}\}$ and d satisfies the Leibniz rule

$$\mathrm{d}(ab) = a\,\mathrm{d}b + \mathrm{d}a\,b.$$

Let in addition $\mathcal{A}$ denote a Hopf algebra and $\Delta_\mathcal{B} : \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ a left $\mathcal{A}$-comodule algebra structure on $\mathcal{B}$. If $\Gamma$ possesses the structure of a left $\mathcal{A}$-comodule

$$\Delta_\Gamma : \Gamma \to \mathcal{A} \otimes \Gamma$$

such that

$$\Delta_\Gamma(a\mathrm{d}b\, c) = (\Delta_\mathcal{B} a)((\mathrm{Id} \otimes \mathrm{d})\Delta_\mathcal{B} b)(\Delta_\mathcal{B} c)$$

then $\Gamma$ is called *left covariant*.

For a family of left covariant FODC $(\Gamma_i, \mathrm{d}_i)_{i=1,\dots,k}$ define $\mathrm{d} = \bigoplus_i \mathrm{d}_i : \mathcal{B} \to \bigoplus_i \Gamma_i$. Then $\Gamma = \mathcal{B}\mathrm{d}\mathcal{B} \subset \oplus_i \Gamma_i$ is a covariant FODC with differential d which is called the *sum* of the calculi $\Gamma_1, \dots, \Gamma_k$.

If $\mathcal{D} \subset \mathcal{B}$ is a subalgebra and $(\Gamma, \mathrm{d})$ is a FODC over $\mathcal{B}$ then $(\Gamma|_\mathcal{D}, \mathrm{d}|_\mathcal{D})$ defined by

$$\Gamma|_\mathcal{D} := \{a\mathrm{d}b \,|\, a, b \in \mathcal{D}\}, \quad \mathrm{d}|_\mathcal{D}(a) := \mathrm{d}a \quad \forall a \in \mathcal{D},$$

is a FODC over $\mathcal{D}$ called the *FODC over $\mathcal{D}$ induced by* $\Gamma$.

### 2.3.2 Higher Order Differential Calculus

A *differential calculus* (DC) over $\mathcal{B}$ is a differential graded algebra $(\Gamma^\wedge = \oplus_{i\in\mathbb{N}_0} \Gamma^{\wedge i}, \mathrm{d})$ such that $\Gamma^{\wedge 0} = \mathcal{B}$ and $\Gamma^\wedge$ is generated by $\mathcal{B}$ and $\mathrm{d}\mathcal{B}$. The product of a DC will usually be denoted by $\wedge$. Assume that $\mathcal{B}$ is a left $\mathcal{A}$-comodule algebra over a Hopf algebra $\mathcal{A}$. Then a DC $(\Gamma^\wedge, \mathrm{d})$ over $\mathcal{B}$ is called (left) *covariant* if $\Gamma^\wedge$ has the structure of a (left) $\mathcal{A}$-comodule algebra such that $(\Gamma^{\wedge 1}, \mathrm{d}|_\mathcal{B})$ is a (left) covariant FODC.

The universal DC of a FODC $(\Gamma, \mathrm{d}_\Gamma)$ over $\mathcal{B}$ is the uniquely determined DC $(\Gamma^\wedge_u, \mathrm{d}_u)$ over $\mathcal{B}$ with $\Gamma^{\wedge 1}_u = \Gamma$ and $\mathrm{d}_u|_\mathcal{B} = \mathrm{d}_\Gamma$ satisfying the following defining property. For any DC $(\Gamma^\wedge, \mathrm{d})$ over $\mathcal{B}$ with $\Gamma^{\wedge 1} = \Gamma$ and $\mathrm{d}|_\mathcal{B} = \mathrm{d}_\Gamma$ there exists a map $\phi : \Gamma^\wedge_u \to \Gamma^\wedge$ of differential graded algebras such that $\phi|_{\mathcal{B}\oplus\Gamma} = \mathrm{Id}$. To construct $(\Gamma^\wedge_u, \mathrm{d}_u)$ consider the tensor algebra $\Gamma^\otimes = \bigoplus_{k=0}^\infty \Gamma^{\otimes k}$ of the $\mathcal{B}$-bimodule $\Gamma$. Then $\Gamma^\wedge_u$ is the quotient of $\Gamma^\otimes$ by the ideal generated by $\{\sum_i \mathrm{d}a_i \otimes \mathrm{d}b_i \,|\, \sum_i a_i \mathrm{d}b_i = 0\}$ and the differential is defined by $\mathrm{d}_u(a_0 \mathrm{d}a_1 \wedge \dots \wedge \mathrm{d}a_n) = \mathrm{d}a_0 \wedge \mathrm{d}a_1 \wedge \dots \wedge \mathrm{d}a_n$.

### 2.3.3 Right Ideals and Quantum Tangent Spaces

From now on we assume $K \subset U$ to be a right coideal subalgebra and $\mathcal{B} \subset \mathcal{A} = U^\circ_\mathcal{C}$ to be the corresponding quantum homogeneous space as in Subsection 2.2.7.

In this situation left covariant first order differential calculi over $\mathcal{B}$ are in one-to-one correspondence to right ideals $\mathcal{R} \subset \mathcal{B}^+$ satisfying $\Delta_\mathcal{B}\mathcal{R} \subset$

$\mathcal{A}\otimes\mathcal{R}+\mathcal{B}^+\mathcal{A}\otimes\mathcal{B}$ [Her02]. The right ideal corresponding to a covariant FODC $\Gamma$ is given by

$$\mathcal{R}=\Big\{\sum_i \varepsilon(a_i)b_i^+\,\Big|\,\sum_i a_i\,\mathrm{d}b_i=0\Big\}\subset\mathcal{B}^+ \tag{16}$$

where $b^+=b-\varepsilon(b)$ for all $b\in\mathcal{B}$. Conversely, to construct the FODC $\Gamma$ corresponding to $\mathcal{R}$ consider the $\mathcal{B}$-bimodule structure on $\tilde{\Gamma}:=\mathcal{A}\otimes(\mathcal{B}^+/\mathcal{R})$ given by

$$c(a\otimes\overline{b})c'=cac'_{(-1)}\otimes\overline{bc'_{(0)}},\qquad c,c'\in\mathcal{B},\ b\in\mathcal{B}^+,\ a\in\mathcal{A} \tag{17}$$

and the differential $\mathrm{d}:\mathcal{B}\to\tilde{\Gamma}$, $\mathrm{d}b=b_{(-1)}\otimes\overline{b^+_{(0)}}$. Then one obtains $\Gamma$ by $\Gamma=\mathrm{Lin}_{\mathbb{C}}\{b_1\,\mathrm{d}b_2\,|\,b_1,b_2\in\mathcal{B}\}$. This implies in particular

$$\sum_i a_i\mathrm{d}b_i=0\ \Leftrightarrow\ \sum_i a_ib_{i(-1)}\otimes b^+_{i(0)}\in\mathcal{A}\otimes\mathcal{R}. \tag{18}$$

To a FODC $\Gamma$ with corresponding right ideal $\mathcal{R}$ one associates the vector space

$$T^\varepsilon_\Gamma=\{f\in\mathcal{B}^*\,|\,f(x)=0\text{ for all }x\in\mathcal{R}\}$$

and the so called *quantum tangent space*

$$T_\Gamma=(T^\varepsilon_\Gamma)^+=\{f\in T^\varepsilon_\Gamma\,|\,f(1)=0\}.$$

The dimension of a first order differential calculus is defined by

$$\dim\Gamma=\dim_{\mathbb{C}}\Gamma/\mathcal{B}^+\Gamma=\dim_{\mathbb{C}}\mathcal{B}^+/\mathcal{R}.$$

Let $\mathcal{B}^\circ$ denote the dual coalgebra of $\mathcal{B}$.

**Proposition 2.5.** [HK03a, Cor. 5] *Let $\mathcal{B}\subset\mathcal{A}$ be as in Subsection 2.2.7. Then there is a canonical one-to-one correspondence between $n$-dimensional left covariant FODC over $\mathcal{B}$ and $(n+1)$-dimensional subspaces $T^\varepsilon\subset\mathcal{B}^\circ$ such that*

$$\varepsilon\in T^\varepsilon,\quad \Delta T^\varepsilon\subset T^\varepsilon\otimes\mathcal{B}^\circ,\quad KT^\varepsilon\subset T^\varepsilon. \tag{19}$$

A covariant FODC $\Gamma\neq\{0\}$ over $\mathcal{B}$ is called *irreducible* if it does not possess any nontrivial quotient (by a left covariant $\mathcal{B}$-bimodule). Note that for finite dimensional calculi this property is equivalent to the property that

$T_\Gamma^\varepsilon$ does not possess any left $K$-invariant right $\mathcal{B}^\circ$-subcomodule $\tilde{T}$ such that $\mathbb{C}\cdot\varepsilon \subsetneq \tilde{T} \subsetneq T_\Gamma^\varepsilon$.

Let $\Gamma$ be a sum of finite dimensional covariant FODC $\Gamma_i$, $i = 1,\dots,N$, over $\mathcal{B}$ with corresponding right ideals $\mathcal{R}_i$. Then the right ideal corresponding to $\Gamma$ is given by $\mathcal{R}_\Gamma = \cap_i \mathcal{R}_{\Gamma_i}$ and therefore the relation $T_\Gamma = T_{\Gamma_1} + \cdots + T_{\Gamma_k}$ of quantum tangent spaces holds. A sum of covariant differential calculi is called a *direct sum* if $\Gamma = \oplus_i \Gamma_i$ is a direct sum of bimodules. This condition is equivalent to $T_\Gamma = \oplus_i T_{\Gamma_i}$.

### 2.3.4 Induced Covariant FODC

Using quantum tangent spaces it is possible to identify induced covariant FODC.

**Proposition 2.6.** [HK03a, Cor. 9] *Let $\mathcal{B} \subset \mathcal{A}$ be as in Subsection 2.2.7 and let $\Gamma$ be a finite dimensional left-covariant FODC over $\mathcal{A}$ with quantum tangent space $T$. Then $\Gamma|_\mathcal{B}$ is finite dimensional if and only if $KT|_\mathcal{B}$ is finite dimensional. In this case the quantum tangent space of $\Gamma|_\mathcal{B}$ coincides with $KT|_\mathcal{B}$.*

**Lemma 2.7.** *Let $\mathcal{B} \subset \mathcal{A}$ be as in Subsection 2.2.7 and let $\Omega$ denote a finite dimensional covariant FODC over $\mathcal{A}$ with corresponding right ideal $\mathcal{R}_\Omega$ and quantum tangent space $T_\Omega$. Assume that the induced FODC $\Gamma$ over $\mathcal{B}$ is finite dimensional with right ideal $\mathcal{R}$ and quantum tangent space $T$. Then the following properties are equivalent:*

*(i)* $T = T_\Omega|_\mathcal{B}$.

*(ii)* $\mathcal{R} = \mathcal{R}_\Omega \cap \mathcal{B}$.

*(iii) The canonical map $\mathcal{A}\otimes_\mathcal{B} \Gamma \to \Omega$, $a\otimes\gamma \mapsto a\gamma$ is injective.*

*Proof.* The canonical map $\mathcal{A}\otimes_\mathcal{B}\Gamma \to \Omega$ of left covariant left $\mathcal{A}$-modules is injective if and only if the restriction

$$ {}^{\mathrm{co}\mathcal{A}}(\mathcal{A}\otimes_\mathcal{B}\Gamma) \to {}^{\mathrm{co}\mathcal{A}}\Omega \tag{20}$$

to the space of left coinvariants is injective. Recall from [Wor89, Thm. 5.1], [HK03a, Lemma 6] that there exist pairings

$$\langle\cdot,\cdot\rangle_\Omega : \Omega\times T_\Omega \to \mathbb{C}, \qquad \langle a\mathrm{d}b, t\rangle_\Omega = \varepsilon(a)t(b), \tag{21}$$

$$\langle\cdot,\cdot\rangle : \Gamma\times T \to \mathbb{C}, \qquad \langle a\mathrm{d}b, t\rangle = \varepsilon(a)t(b), \tag{22}$$

which induce nondegenerate pairings

$$
{}^{\mathrm{co}\mathcal{A}}\Omega \times T_\Omega \to \mathbb{C}, \qquad {}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma) \times T \to \mathbb{C}. \tag{23}
$$

Now, (20) is injective if and only if $T_\Omega$ separates ${}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma)$. In view of the nondegeneracy of the second pairing in (23) and Proposition 2.6 the latter is equivalent to $T = T_\Omega|_{\mathcal{B}}$. Therefore (i) is equivalent to (iii). The equivalence between (i) and (ii) holds by duality. □

### 2.3.5 Determining $\Gamma_u^{\wedge 2}$

Quantum tangent spaces can also be employed to obtain information about higher order differential calculi. Let $\Omega$, $\Gamma$, $T_\Omega$, and $T$ be as in Lemma 2.7. In analogy to (21), (22) there exists a pairing

$$
\begin{aligned}
&\langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathcal{A} \otimes_{\mathcal{B}} \Gamma \otimes_{\mathcal{B}} \Gamma \times T_\Omega \otimes T \to \mathbb{C},\\
&\langle\!\langle a \otimes \rho \otimes \zeta, s \otimes t \rangle\!\rangle := \varepsilon(a) \langle \rho, s^+_{(0)} \rangle_\Omega \, s_{(1)}(\zeta_{(-1)}) \langle \zeta_{(0)}, t \rangle.
\end{aligned} \tag{24}
$$

In particular

$$
\langle\!\langle a \otimes \mathrm{d}b \otimes \mathrm{d}c, s \otimes t \rangle\!\rangle = \varepsilon(a) s(b^+ c_{(-1)}) t(c_{(0)}). \tag{25}
$$

To verify that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is well defined note that

$$
\langle a\mathrm{d}b\, c, s \rangle_\Omega = \langle a\mathrm{d}(bc) - ab\mathrm{d}c, s \rangle_\Omega = \varepsilon(a) s(b^+ c) = \langle a\mathrm{d}b, s^+_{(0)} \rangle_\Omega s_{(1)}(c)
$$

and therefore

$$
\begin{aligned}
\langle \rho c, s^+_{(0)} \rangle_\Omega \, s_{(1)}(\zeta_{(-1)}) \langle \zeta_{(0)}, t \rangle &= \langle \rho, s^+_{(0)} \rangle_\Omega \, s_{(1)}(c) s_{(2)}(\zeta_{(-1)}) \langle \zeta_{(0)}, t \rangle\\
&= \langle \rho, s^+_{(0)} \rangle_\Omega \, s_{(1)}(c_{(-1)} \zeta_{(-1)}) \langle c_{(0)} \zeta_{(0)}, t \rangle.
\end{aligned}
$$

**Lemma 2.8.** *Let $\mathcal{B} \subset \mathcal{A}$ be as in Subsection 2.2.7 and let $\Omega$, $\Gamma$, $T_\Omega$, and $T$ be as in Lemma 2.7. Assume $T = T_\Omega|_{\mathcal{B}}$. Then the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ induces a nondegenerate pairing*

$$
{}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma \otimes_{\mathcal{B}} \Gamma) \times (T_\Omega \otimes T)/T_0 \to \mathbb{C}, \tag{26}
$$

*where $T_0 = \{\sum_i s_i \otimes t_i \in T_\Omega \otimes T \mid \sum_i s^+_{i(0)}|_{\mathcal{B}} \otimes s_{i(1)} t_i = 0$ in $\mathcal{B}^\circ \otimes \mathcal{B}^\circ\}$.*

*Proof.* Note first that

$$
\langle\!\langle a \otimes \mathrm{d}b \otimes \mathrm{d}c, s \otimes t \rangle\!\rangle = \langle\!\langle \kappa(a_{(1)} b_{(-1)} c_{(-1)}) a_{(2)} \otimes \mathrm{d}b_{(0)} \otimes \mathrm{d}c_{(0)}, s \otimes t \rangle\!\rangle.
$$

By definition (24) the pairing (26) is well defined and by (25) the elements of $(T_\Omega \otimes T)/T_0$ are separated by ${}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma \otimes_{\mathcal{B}} \Gamma)$.

Conversely, recall that $\Gamma$ is a projective left $\mathcal{B}$-module by Remark 2.4(iii). By Lemma 2.7 one obtains a canonical inclusion

$$\mathcal{A} \otimes_{\mathcal{B}} \Gamma \otimes_{\mathcal{B}} \Gamma \subset \Omega \otimes_{\mathcal{B}} \Gamma \cong \Omega \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{B}} \Gamma \subset \Omega \otimes_{\mathcal{A}} \Omega.$$

Via this inclusion one identifies $\kappa(a_{(-1)}b_{(-1)}) \otimes \mathrm{d}a_{(0)} \otimes {}^{\mathrm{co}\mathcal{A}} \mathrm{d}b_{(0)} \in (\mathcal{A} \otimes_{\mathcal{B}} \Gamma \otimes_{\mathcal{B}} \Gamma)$ with $\kappa(b_{(-1)})\omega_L(a) \otimes \mathrm{d}b_{(0)} \in \Omega \otimes_{\mathcal{B}} \Gamma$ and with $\omega_L(a^+ b_{(-1)}) \otimes \omega_L(b_{(0)}) \in \Omega \otimes_{\mathcal{A}} \Omega$ where $\omega_L(a) = \kappa(a_{(1)})\mathrm{d}a_{(2)}$ for all $a \in \mathcal{A}$.

Recall [Wor89, p.164] that the pairing

$$\begin{aligned} \langle\!\langle \cdot, \cdot \rangle\!\rangle : {}^{\mathrm{co}\mathcal{A}}(\Omega \otimes \Omega) \times (T_\Omega \otimes_{\mathbb{C}} T_\Omega) \to \mathbb{C}, \\ \langle\!\langle \omega_L(a) \otimes \omega_L(b), s \otimes t \rangle\!\rangle = s(a)t(b), \quad a, b \in \mathcal{A} \end{aligned}$$

is nondegenerate and compatible with (26). Therefore

$$\left\langle\!\!\left\langle \sum_i \kappa(a_{i(-1)}b_{i(-1)}) \otimes \mathrm{d}a_{i(0)} \otimes \mathrm{d}b_{i(0)}, s \otimes t \right\rangle\!\!\right\rangle = 0 \quad \text{for all } s \in T_\Omega,\ t \in T$$

implies $\langle\!\langle \sum_i \omega_L(a_i^+ b_{i(-1)}) \otimes \omega_L(b_{i(0)}), s \otimes t \rangle\!\rangle = 0$ for all $s, t \in T_\Omega$, and hence

$$\sum_i \omega_L(a_i^+ b_{i(-1)}) \otimes \omega_L(b_{i(0)}) = 0.$$

□

**Corollary 2.9.** *Let $\mathcal{B} \subset \mathcal{A}$ be as in Subsection 2.2.7 and let $\Omega$, $\Gamma$, $T_\Omega$, $T$, and $T_0$ be as in Lemma 2.8. Assume that $\Gamma\mathcal{B}^+ \subset \mathcal{B}^+\Gamma$. Then*

$$\dim_{\mathbb{C}} T_0 = \dim_{\mathbb{C}} T(\dim_{\mathbb{C}} T_\Omega - \dim_{\mathbb{C}} T).$$

*Proof.* By Lemma 2.8 and the Remark 2.4(i) one gets

$$\dim_{\mathbb{C}}(T_\Omega \otimes T)/T_0 = \dim_{\mathbb{C}}({}^{\mathrm{co}\mathcal{A}}\mathcal{A} \otimes_{\mathcal{B}} \Gamma \otimes_{\mathcal{B}} \Gamma) = \dim_{\mathbb{C}}(\Gamma \otimes_{\mathcal{B}} \Gamma)/(\mathcal{B}^+\Gamma \otimes_{\mathcal{B}} \Gamma).$$

The inclusion $\Gamma\mathcal{B}^+ \subset \mathcal{B}^+\Gamma$ implies that the canonical map

$$(\Gamma \otimes_{\mathcal{B}} \Gamma)/(\mathcal{B}^+\Gamma \otimes_{\mathcal{B}} \Gamma) \to \Gamma/\mathcal{B}^+\Gamma \otimes_{\mathbb{C}} \Gamma/\mathcal{B}^+\Gamma$$

is an isomorphism. Therefore $\dim_{\mathbb{C}} T_0 = (\dim_{\mathbb{C}} T_\Omega)(\dim_{\mathbb{C}} T) - (\dim \Gamma)^2$. □

**Corollary 2.10.** *Let $\mathcal{B} \subset \mathcal{A}$ be as in Subsection 2.2.7 and let $\Omega$, $\Gamma$, $T_\Omega$, $T$, and $T_0$ be as in Lemma 2.8. Then the pairing (26) induces a pairing*

$$(\mathcal{A} \otimes_{\mathcal{B}} \Gamma_u^{\wedge 2}) \times \mathrm{Lin}_{\mathbb{C}} \left\{ \sum_i s_i \otimes t_i \in T_\Omega \otimes T \;\middle|\; \sum_i s_i t_i \in T \right\} \Big/ T_0 \to \mathbb{C} \tag{27}$$

*which is nondegenerate when restricted to ${}^{\mathrm{co}\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} \Gamma_u^{\wedge 2})$ in the first component.*

*Proof.* Note first that $\Gamma_{\mathrm{u}}^{\wedge 2} = \Gamma^{\otimes 2}/\Lambda$ where $\Lambda \subset \Gamma^{\otimes 2}$ is the *left* $\mathcal{B}$-submodule generated by $\{\sum_i \mathrm{d}a_i \otimes \mathrm{d}b_i \in \Gamma^{\otimes 2} \,|\, \sum_i a_i \mathrm{d}b_i = 0\}$. It suffices to show that with respect to the pairing (24) one has

$$(\mathcal{A} \otimes_{\mathcal{B}} \Lambda)^{\perp} = \left\{ \sum_i s_i \otimes t_i \in T_\Omega \otimes T \,\middle|\, \sum_i s_i t_i \in T \right\}.$$

Assume that $\sum_i a_i \mathrm{d}b_i = 0$. Then

$$\begin{aligned}
\sum_{i,j} \langle\!\langle 1 \otimes \mathrm{d}a_i \otimes \mathrm{d}b_i, s_j \otimes t_j \rangle\!\rangle &= \sum_{i,j} s_j(a_i^+ b_{i(-1)}) t_j(b_{i(0)}) \\
&\overset{(18)}{=} -\sum_{i,j} \varepsilon(a_i) s_j(b_{i(-1)}) t_j(b_{i(0)}) \\
&= -\sum_{i,j} s_j t_j(\varepsilon(a_i) b_i) = -\sum_{i,j} s_j t_j(\varepsilon(a_i) b_i^+).
\end{aligned}$$

Hence (16) implies that $\sum_j s_j \otimes t_j \in (\mathcal{A} \otimes_{\mathcal{B}} \Lambda)^{\perp}$ if and only if $\sum_j s_j t_j \in T$. $\square$

# 3 Differential Calculus on Quantized Irreducible Flag Manifolds

In the previous Section we have recalled basic notions and developed the general theory necessary for the investigation of covariant DC on quantum homogeneous spaces. Now we turn to the concrete example of quantized flag manifolds. We first collect some facts about the corresponding algebras. Then using the tools from the previous section the canonical covariant DC over irreducible quantized flag manifolds is constructed and investigated in detail.

## 3.1 Quantized Flag Manifolds

### 3.1.1 Homogeneous Coordinate Rings

The quantized homogeneous coordinate ring $S_q[G/P_S]$ of a generalized flag manifold $G/P_S$ is defined to be the subalgebra of $\mathbb{C}_q[G]$ generated by the matrix coefficients $\{c^{\lambda}_{f,v_\lambda} \,|\, f \in V(\lambda)^*\}$, [CP94], [LR92], [TT91], [Soi92], where $v_\lambda$ is a highest weight vector of $V(\lambda)$. As a $U_q(\mathfrak{g})$-module algebra $S_q[G/P_S]$ is isomorphic to $\bigoplus_{n=0}^{\infty} V(n\lambda)^*$, where $\lambda = \sum_{s \notin S} \omega_s$, endowed with the Cartan multiplication

$$V(n_1\lambda)^* \otimes V(n_2\lambda)^* \to V((n_1+n_2)\lambda)^*.$$

Recall that the subspace $V(2\lambda)\subset V(\lambda)\otimes V(\lambda)$ is the eigenspace of $\hat{R}_{\lambda,\lambda}$ with corresponding eigenvalue $q^{(\lambda,\lambda)}$. It is known ([TT91], [Bra94]) that $S_q[G/P_S]$ is quadratic, more explicitly

$$S_q[G/P_S]\cong \mathbb{C}\big\langle f_1,\dots,f_N\big\rangle/\big(\sum_{k,l\in I}\hat{R}^{ij}_{kl}f_kf_l-q^{(\lambda,\lambda)}f_if_j\big)$$

where $\hat{R}$, $N$, $I$ are as in Section 2.2.5. Similarly the dual quantized homogeneous coordinate ring $S_q[G/P_S^{\mathrm{op}}]$ of $G/P_S$ is defined to be the subalgebra of $\mathbb{C}_q[G]$ generated by $\{c^{-w_0\lambda}_{v,f_{-\lambda}}\,|\,v\in V(\lambda)\}$ where $f_{-\lambda}\in V(-w_0\lambda)\cong V(\lambda)^*$ denotes the lowest weight vector dual to $v_\lambda$. In terms of generators and relations one has

$$S_q[G/P_S^{\mathrm{op}}]\cong \mathbb{C}\big\langle v_1,\dots,v_N\big\rangle/\big(\sum_{k,l\in I}\check{R}^{ij}_{kl}v_kv_l-q^{(\lambda,\lambda)}v_iv_j\big).$$

The $U_q(\mathfrak{g})$-module structure of $S_q[G/P_S]$ and $S_q[G/P_S^{\mathrm{op}}]$ is given by identifying the generators $\{f_i\,|\,i\in I\}$ and $\{v_i\,|\,i\in I\}$ with the bases of $V(\lambda)^*$ and $V(\lambda)$ chosen in Section 2.2.5. For notational reasons in what follows suppose that $v_\lambda=v_N$ and $f_\lambda=f_N$.

### 3.1.2 The Subalgebra $S_q[G/P_S]^{c=1}_{\mathbb{C}}\subset\mathbb{C}_q[G]$

The tensor product $S_q[G/P_S]_{\mathbb{C}}:=S_q[G/P_S]\otimes S_q[G/P_S^{\mathrm{op}}]$ can be endowed with a $U_q(\mathfrak{g})$-module algebra structure by

$$v_if_j:=q^{(\lambda,\lambda)}\sum_{k,l\in I}\grave{R}^{-ij}_{\ \ kl}f_k\otimes v_l. \tag{28}$$

To simplify notation the tensor product symbol will be omitted in the following. The algebra $S_q[G/P_S]_{\mathbb{C}}$ admits a character $\varepsilon$ defined by $\varepsilon(v_i)=\varepsilon(f_i)=\delta_{iN}$. Note that

$$c:=\sum_{i\in I}v_if_i=q^{(\lambda,\lambda)}\sum_{i,k,l\in I}\grave{R}^{-ii}_{\ \ kl}f_kv_l$$

is a central invariant element of $S_q[G/P_S]_{\mathbb{C}}$ with $\varepsilon(c)=1$. The quotient $S_q[G/P_S]^{c=1}_{\mathbb{C}}:=S_q[G/P_S]_{\mathbb{C}}/(c-1)$ is $\mathbb{Z}$-graded by $\deg f_i=1$, $\deg v_i=-1$. Let $S^n_q[G/P_S]^{c=1}_{\mathbb{C}}\subset S_q[G/P_S]^{c=1}_{\mathbb{C}}$ denote the homogeneous component of degree $n$ with respect to this grading.

**Lemma 3.1.** *The $U_q(\mathfrak{g})$-module algebra $S_q[G/P_S]^{c=1}_{\mathbb{C}}$ is isomorphic to the $U_q(\mathfrak{g})$-module subalgebra of $\mathbb{C}_q[G]$ generated by the matrix coefficients $c^\lambda_{f,v_N}$, $c^{-w_0\lambda}_{v,f_N}$, $f\in V(\lambda)^*$, $v\in V(\lambda)$. The isomorphism is given by*

$$f\mapsto c^\lambda_{f,v_N}\qquad v\mapsto c^{-w_0\lambda}_{v,f_N}.$$

*Proof.* The torus $\mathbb{C}[K_i, K_i^{-1} \mid i = 1, \ldots, r] \subset U_q(\mathfrak{g})$ acts on $\mathbb{C}_q[G]$ by

$$K_i \triangleright c^\lambda_{f,v} = c^\lambda_{f,K_i v}.$$

The eigenspace decomposition with respect to this action induces a $\mathbb{Z}$-grading on the subalgebra $A \subset \mathbb{C}_q[G]$ generated by the matrix coefficients $c^\lambda_{f,v_N}$, $c^{-w_0\lambda}_{v,f_N}$, $f \in V(\lambda)^*$, $v \in V(\lambda)$. More precisely, $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where

$$A_n = \mathrm{Lin}_\mathbb{C}\{c^{k\lambda}_{f,v_N^{\otimes k}} c^{-lw_0\lambda}_{v,f_N^{\otimes l}} \mid k, l \geq 0, k - l = n\}.$$

Note that $\mathrm{Lin}_\mathbb{C}\{c^\mu_{f,v_\mu} c^\nu_{g,v_{w_0\nu}} \mid f \in V(\mu)^*, g \in V(\nu)^*\} \cong V(\mu)^* \otimes V(\nu)^*$ where $v_\mu$ and $v_{w_0\nu}$ denote a highest weight vector of $V(\mu)$ and a lowest weight vector of $V(\nu)^*$, respectively. Therefore the relation $q^{(\lambda,\lambda)} \sum_{i,k,l\in I} (\grave{R}^-)^{ii}_{kl} c^\lambda_{f_k,v_N} c^{-w_0\lambda}_{v_l,f_N} = 1$ implies that $A_n$ can be written as a direct limit

$$A_n \cong \lim_{k\to\infty} V(k\lambda)^* \otimes V((k-n)\lambda).$$

of vector spaces. By construction the homogeneous components $S^n_q[G/P_S]^{c=1}_\mathbb{C}$ of

$$S_q[G/P_S]^{c=1}_\mathbb{C} = \bigoplus_{n=0}^{\infty} S^n_q[G/P_S]^{c=1}_\mathbb{C}$$

allow the same presentation. $\square$

### 3.1.3 Quantized Flag Manifolds in Terms of Generators and Relations

Lemma 3.1 implies that the subalgebra $S^0_q[G/P_S]^{c=1}_\mathbb{C}$ is isomorphic to the subalgebra $\mathbb{A}^q_\lambda \subset \mathbb{C}_q[G]$ generated by the elements $z_{ij} := c^\lambda_{f_i,v_N} c^{-w_0\lambda}_{v_j,f_N}$. It follows from

$$c^{-w_0\lambda}_{v_i,f_N} c^\lambda_{f_j,v_N} = q^{(\lambda,\lambda)} \sum_{kl\in I} (\grave{R}^-)^{ij}_{kl} c^\lambda_{f_k,v_N} c^{-w_0\lambda}_{v_l,f_N} \tag{29}$$

that the following relations hold in $\mathbb{A}^q_\lambda$:

$$\hat{P}_{12}\acute{R}_{23} zz = 0, \quad \check{P}_{34}\acute{R}_{23} zz = 0, \quad q^{(\lambda,\lambda)} \sum_{i,j\in I} C_{ij} z_{ij} = 1, \tag{30}$$

where $\hat{P} := (\hat{R} - q^{(\lambda,\lambda)}\mathrm{Id})$, $\check{P} := (\check{R} - q^{(\lambda,\lambda)}\mathrm{Id})$, $C_{kl} := \sum_{i\in I} (\grave{R}^-)^{ii}_{kl}$, and leg-notation is applied in the first two formulae. Thus, explicitly written the first two equations of (30) take the form

$$\sum_{m,n,p,t\in I} \hat{P}^{ij}_{nm} \acute{R}^{mk}_{pt} z_{np} z_{tl} = 0, \qquad \sum_{m,n,p,t\in I} \check{P}^{kl}_{mt} \acute{R}^{jm}_{np} z_{in} z_{pt} = 0.$$

Let $\tilde{\mathbb{A}}^q_\lambda$ denote the free algebra $\mathbb{C}\langle z_{ij}\rangle$ divided by the ideal generated by the relations (30). It follows from the Yang-Baxter-Equation that the left $U_q(\mathfrak{g})$-module homomorphisms

$$V(n\lambda)^* \otimes V(n\lambda) \to \tilde{\mathbb{A}}^q_\lambda, \quad f_{i_1}\dots f_{i_n} v_{j_1}\dots v_{j_n} \mapsto$$
$$q^{-n(n-1)(\lambda,\lambda)/2} \sum_{\substack{k_1,\dots,k_n\\ l_1,\dots,l_n}} \left( \prod_{k=1}^{n-1} \prod_{l=1}^{n-k} \acute{R}_{n+k-l,n+k-l+1} \right)^{i_1\dots i_n j_q \dots j_n}_{k_1 l_1 \dots k_n l_n} z_{k_1 l_1}\dots z_{k_n l_n}$$

$$V(n\lambda)^* \otimes V(n\lambda) \hookrightarrow V((n+1)\lambda)^* \otimes V((n+1)\lambda), \quad f_{i_1}\dots f_{i_n} v_{j_1}\dots v_{j_n} \mapsto$$
$$q^{(n+1)(\lambda,\lambda)} \sum_{\substack{k_1,\dots,k_n\\ i,l,m}} \left( \prod_{k=1}^{n+1} \grave{R}^-{}_{k,k+1} \right)^{i i i_1 \dots i_n}_{m k_1 \dots k_n l} f_m f_{k_1}\dots f_{k_n} v_l v_{j_1}\dots v_{j_n}$$

are well defined. Thus one obtains a surjection

$$A_0 \cong \varinjlim V(n\lambda)^* \otimes V(n\lambda) \to \tilde{\mathbb{A}}^q_\lambda.$$

Note that since $A_0 \subset \mathbb{C}_q[G]$ the isotypical components of the $U_q(\mathfrak{g})$-module $\varinjlim V(n\lambda)^* \otimes V(n\lambda)$ are finite dimensional. As the homomorphism $\tilde{\mathbb{A}}^q_\lambda \to \mathbb{A}^q_\lambda$, $z_{ij} \mapsto z_{ij}$ is surjective and $\mathbb{A}^q_\lambda \cong \varinjlim V(n\lambda)^* \otimes V(n\lambda)$ this yields $\tilde{\mathbb{A}}^q_\lambda \cong \mathbb{A}^q_\lambda$. Define

$$\mathbb{C}_q[G/L_S] = \{a \in \mathbb{C}_q[G] \,|\, a_{(1)}\, a_{(2)}(k) = \varepsilon(k)a \quad \forall k \in K\}, \tag{31}$$

where $K := U_q(\mathfrak{l}_S)$ is the Hopf subalgebra of $U_q(\mathfrak{g})$ generated by the elements $\{K_i, K_i^{-1}, E_j, F_j \,|\, j \in S,\, i = 1,\dots,r\}$. By construction $\mathbb{C}_q[G/L_S]$ is a left $\mathbb{C}_q[G]$-comodule algebra containing $\mathbb{A}^q_\lambda$. The following proposition was proved in [Sto02], [HK03b].

**Proposition 3.2.** $\mathbb{A}^q_\lambda \cong \mathbb{C}_q[G/L_S]$ *as left* $\mathbb{C}_q[G]$*-comodule algebras.*

## 3.2 First Order Differential Calculus over $\mathbb{C}_q[G/L_S]$

### 3.2.1 Notations and Conventions

From now on we assume that $G/P_S$ is an irreducible flag manifold, in particular $S = \{\alpha_s\}$, $\lambda = \omega_s$ for a fixed $s \in \{1,\dots,r\}$. To simplify notation, the isomorphic $\mathbb{C}_q[G]$-comodule algebras $\mathbb{C}_q[G/L_S]$, $\mathbb{A}^q_\lambda$, and $\tilde{\mathbb{A}}^q_\lambda$ will be denoted by $\mathcal{B}$. By definition of $\mathbb{C}_q[G/L_S]$ the algebra $\mathcal{B}$ is a quantum homogeneous spaces in the sense of 2.2.7. Further, define $I_{(1)} := \{i \in I \,|\, (\omega_s, \omega_s - \alpha_s - \mathrm{wt}(v_i)) =$

$0\}$. Note that the elements of $I_{(1)}$ label a basis of the $U_q(\mathfrak{l}_S)$-submodules $V(\omega_s)_{(1)} := \mathrm{Lin}_{\mathbb{C}}\{v \in V(\omega_s) \,|\, \mathrm{wt}(v) = \omega_s - \alpha_s - \beta, \, \beta \in Q, (\omega_s, \beta) = 0\} \subset V(\omega_s)$ and $V(\omega_s)^*_{(1)} := \mathrm{Lin}_{\mathbb{C}}\{f \in V(\omega_s)^* \,|\, \mathrm{wt}(f) = -\omega_s + \alpha_s + \beta, \, \beta \in Q, (\omega_s, \beta) = 0\} \subset V(\omega_s)^*$. As in 3.1.1 assume that $v_N = v_{\omega_s}$ is the highest weight vector of $V(\omega_s)$ in the basis $\{v_i \,|\, i \in I\}$ chosen in 2.2.5.

Fix a reduced decomposition of the longest element of the Weyl group. Let $E_\beta, F_\beta$, $\beta \in R^+$, denote the corresponding root vectors in $U$ [CP94, 8.1], [KS97, 6.2].

Finally we introduce the abbreviation $M := \dim_{\mathbb{C}} \mathfrak{g}/\mathfrak{p}_S = \#\overline{R^+_S}$.

### 3.2.2 FODC over $S_q[G/P_S]$

In analogy to the construction of $\mathcal{B}$ via $S_q[G/P_S]$ one can obtain covariant FODC over $\mathcal{B}$ by first constructing covariant FODC over $S_q[G/P_S]$. Consider the left $S_q[G/P_S]$-module $\Gamma_+$ generated by elements $\mathrm{d}f_i$, $i \in I$, and relations

$$\sum_{i,j\in I} a_{ij} f_i \mathrm{d} f_j = 0 \quad \text{if } \sum_{i,j\in I} a_{ij} f_i \otimes f_j \in V(\mu)^* \subset V(\omega_s)^* \otimes V(\omega_s)^*, \ \mu \neq 2\omega_s, 2\omega_s - \alpha_s.$$

As the subspaces $V(2\omega_s)$ and $V(2\omega_s - \alpha_s)$ of $V(\omega_s) \otimes V(\omega_s)$ are uniquely determined by the respective eigenvalues $q^{(\omega_s,\omega_s)}$ and $-q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)}$ of $\hat{R}_{\omega_s,\omega_s}$ these relations are equivalent to

$$\sum_{i,j\in I} \left[\hat{P}\hat{Q}\right]^{kl}_{ij} f_i \mathrm{d} f_j = 0 \qquad \forall k, l \in I \tag{32}$$

where as above $\hat{P} = (\hat{R} - q^{(\omega_s,\omega_s)}\mathrm{Id})$ and $\hat{Q} := (\hat{R} + q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)}\mathrm{Id})$. The left module $\Gamma_+$ can be endowed with an $S_q[G/P_S]$-bimodule structure by

$$\mathrm{d} f_i f_j = q^{(\alpha_s,\alpha_s)-(\omega_s,\omega_s)} \sum_{k,l\in I} \hat{R}^{ij}_{kl} f_k \mathrm{d} f_l. \tag{33}$$

Indeed, it follows from the Yang-Baxter-Equation for $\hat{R}$ that this right module structure is well defined on $\Gamma_+$. As $\mathrm{Lin}_{\mathbb{C}}\{\mathrm{d}f_i\} \cong V(\omega_s)^*$ the bimodule $\Gamma_+$ inherits a $U_q(\mathfrak{g})$-module structure.

Define a linear map

$$\mathrm{d} : S_q[G/P_S] \to \Gamma_+$$

by $\mathrm{d}(f_i) := \mathrm{d}f_i$ and $\mathrm{d}(ab) = \mathrm{d}a\, b + a\mathrm{d}b$ for all $a, b \in S_q[G/P_S]$. To verify that $\mathrm{d}$ is well defined note first that for $\sum_{i,j\in I} a_{ij} f_i \otimes f_j \in V(\mu)^* \subset V(\omega_s)^* \otimes V(\omega_s)^*$,

$\mu \neq 2\omega_s, 2\omega_s - \alpha_s$, one has $\sum_{i,j\in I} a_{ij} f_i \mathrm{d}f_j = \sum_{i,j\in I} a_{ij}\mathrm{d}f_i\, f_j = 0$. Moreover, if $\sum_{i,j\in I} a_{ij} f_i \otimes f_j \in V(2\omega_s - \alpha_s)^* \subset V(\omega_s)^* \otimes V(\omega_s)^*$ then

$$\sum_{i,j,k,l\in I} a_{ij} \hat{Q}^{ij}_{kl} f_k \otimes f_l = 0$$

and therefore

$$\sum_{i,j\in I} a_{ij}\mathrm{d}f_i\, f_j + a_{ij} f_i \mathrm{d}f_j = \sum_{i,j,k,l\in I} a_{ij} \left( q^{(\alpha_s,\alpha_s)-(\omega_s,\omega_s)} \hat{R} + \mathrm{Id} \right)^{ij}_{kl} f_k \mathrm{d}f_l = 0.$$

### 3.2.3 FODC over $S_q[G/P_S]_{\mathbb{C}}$

One can use $\Gamma_+$ to construct a covariant FODC $(\Gamma_{+,\mathbb{C}}, \partial)$ over $S_q[G/P_S]_{\mathbb{C}}$ as follows. The left $S_q[G/P_S]_{\mathbb{C}}$-module

$$\Gamma_{+,\mathbb{C}} := S_q[G/P_S]_{\mathbb{C}} \otimes_{S_q[G/P_S]} \Gamma_+ \cong S_q[G/P_S^{\mathrm{op}}] \otimes_{\mathbb{C}} \Gamma_+$$

can be endowed with a right $S_q[G/P_S]_{\mathbb{C}}$ module structure by

$$\mathrm{d}f_i v_j = q^{-(\omega_s,\omega_s)} \sum_{k,l\in I} \acute{R}^{ij}_{kl} v_k \mathrm{d}f_l. \tag{34}$$

The differential $\partial : S_q[G/P_S]_{\mathbb{C}} \to \Gamma_{+,\mathbb{C}}$ defined by

$$\partial(v_i) = 0, \quad \partial(f_i) = \mathrm{d}f_i$$

and Leibniz rule is well defined in view of (28), (34).

There exists a pairing

$$\begin{aligned} &\langle \cdot,\cdot \rangle : \Gamma_{+,\mathbb{C}} \otimes V(\omega_s)_{(1)} \to \mathbb{C}, \\ &\langle w\mathrm{d}f, v \rangle := \varepsilon(w) f(v) \quad \text{ for } w \in S_q[G/P_S]_{\mathbb{C}},\ f \in S_q[G/P_S] \end{aligned} \tag{35}$$

where $\varepsilon$ and $V(\omega_s)_{(1)}$ have been defined in 3.1.2 and 3.2.1, respectively. To verify that $\langle \cdot,\cdot \rangle$ is well defined note that $\langle f_i \mathrm{d}f_j, v \rangle \neq 0$ implies $\mathrm{wt}(f_i) = -\omega_s$, $\mathrm{wt}(f_j) = -\omega_s + \alpha_s + \beta$, $(\beta, \omega_s) = 0$, but then $f_i \otimes f_j \in V(2\omega_s)^* \oplus V(2\omega_s - \alpha_s)^* \subset V(\omega_s)^* \otimes V(\omega_s)^*$.

Equations (28), (32), (33), and (34) imply

$$\partial c\, v_k = v_k \partial c, \quad \partial c\, f_k = q^{(\alpha_s,\alpha_s)} f_k \partial c, \quad \mathrm{d}f_k\, c = c\mathrm{d}f_k + (q^{(\alpha_s,\alpha_s)} - 1) f_k \partial c. \tag{36}$$

### 3.2.4 The FODC $\Gamma_\partial$

Let $\Lambda \subset \Gamma_{+,\mathbb{C}}$ denote the subbimodule generated by $\partial c$, $(c-1)\Gamma_{+,\mathbb{C}}$, and $\Gamma_{+,\mathbb{C}}(c-1)$. Then $\Gamma_{+,\mathbb{C}}/\Lambda$ is a covariant FODC over $S_q[G/P_S]_{\mathbb{C}}^{c=1}$ which by (36) as a left module is generated by $\mathrm{d}f_i$, $i \in I$, and relations (32) and $\partial c = 0$. As $\varepsilon(v_i) \neq 0$ if and only if $v_i = v_N$ and $f_N(v) = 0$ for all $v \in V(\omega_s)_{(1)}$ one obtains $\langle \partial c, v\rangle = 0$ for all $v \in V(\omega_s)_{(1)}$. Therefore the pairing

$$\langle\,,\,\rangle : \Gamma_{+,\mathbb{C}}/\Lambda \otimes V(\omega_s)_{(1)} \to \mathbb{C} \tag{37}$$

induced by (35) is well defined. Let $\Gamma_\partial \subset \Gamma_{+,\mathbb{C}}/\Lambda$ denote the FODC over $\mathcal{B} \subset S_q[G/P_S]_{\mathbb{C}}^{c=1}$ induced by $\Gamma_{+,\mathbb{C}}/\Lambda$.

**Proposition 3.3.** *(i) As a left $\mathcal{B}$-module $\Gamma_\partial$ is generated by the differentials $\partial z_{ij}$, $i,j \in I$, and relations*

$$\hat{P}_{12}\hat{Q}_{12}\acute{R}_{23}z\partial z = 0, \tag{38}$$

$$\check{P}_{34}\acute{R}_{23}z\partial z = 0, \tag{39}$$

$$\sum_{i,j\in I} C_{ij}\partial z_{ij} = 0. \tag{40}$$

*(ii) The right $\mathcal{B}$-module structure of $\Gamma_\partial$ is given by*

$$\partial z z = q^{(\alpha_s,\alpha_s)}\grave{R}^{-}_{23}\hat{R}_{12}\check{R}^{-}_{34}\acute{R}_{23}z\partial z. \tag{41}$$

*(iii)* $\dim \Gamma_\partial = M$.

*(iv)* $\Gamma_\partial \mathcal{B}^+ = \mathcal{B}^+\Gamma_\partial$.

*(v) The quantum tangent space of $\Gamma_\partial$ is* $\mathrm{Lin}_{\mathbb{C}}\{F_\beta \,|\, \beta \in \overline{R_S^+}\}$.

*Proof.* Note first that the relations (38)-(41) hold by construction in the $\mathcal{B}$-bimodule $\Gamma_\partial$. Thus, by (41) $\Gamma_\partial$ is generated by $\{\partial z_{ij}\,|\, i,j \in I\}$ as a left $\mathcal{B}$-module. Moreover, by (34) the restriction of the pairing (37) to $\mathrm{Lin}_{\mathbb{C}}\{\partial z_{iN} \,|\, i \in I_{(1)}\} \times V(\omega_s)_{(1)}$ is nondegenerate. Therefore one obtains $\dim \Gamma_\partial \geq \dim_{\mathbb{C}} V(\omega_s)_{(1)} = M$.

To prove (i)-(iii) consider the left $\mathcal{B}$-module $\Gamma'_\partial$ generated by elements $\partial z_{ij}$, $i,j \in I$, and relations (38)-(40). By the categorial equivalence Theorem 2.3 it suffices to verify that $\dim_{\mathbb{C}} \Gamma'_\partial/\mathcal{B}^+\Gamma'_\partial \leq M$. To this end note first that (39) multiplied by $\check{R}^{-}_{34}$, the relation $\varepsilon(z_{ij}) = \delta_{iN}\delta_{jN}$, and (10) imply

$$\sum_{m,n\in I} \acute{R}^{jk}_{mn} z_{im}\partial z_{nl} \in \mathcal{B}^+\Gamma'_\partial \quad \text{for all } i,j,k,l \in I \text{ such that } l \neq N.$$

In particular one obtains

$$\partial z_{kl} \in \mathcal{B}^+\Gamma'_\partial \quad \text{for all } k,l \in I \text{ such that } l \neq N. \tag{42}$$

Moreover, one calculates

$$\begin{aligned}
&q^{(\omega_s,\omega_s)} C_{34} \grave{R}^-_{23} \hat{R}_{12} \check{R}^-_{34} \acute{R}_{23} z\partial z \\
&\qquad \overset{(39)}{=} C_{34} \grave{R}^-_{23} \hat{R}_{12} \acute{R}_{23} z\partial z \\
&\qquad \overset{(38)}{=} q^{2(\omega_s,\omega_s)-(\alpha_s,\alpha_s)} C_{34} \grave{R}^-_{23} \hat{R}^-_{12} \acute{R}_{23} z\partial z + q^{(\omega_s,\omega_s)}(1 - q^{-(\alpha_s,\alpha_s)}) \underbrace{C_{34} z\partial z}_{=0 \text{ by } (40)} \\
&\qquad = q^{2(\omega_s,\omega_s)-(\alpha_s,\alpha_s)} C_{12} \grave{R}^-_{23} \check{R}^-_{34} \acute{R}_{23} z\partial z \\
&\qquad \overset{(39)}{=} q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)} C_{12} z\partial z \\
&\qquad \overset{(30)}{=} q^{-(\alpha_s,\alpha_s)} \partial z.
\end{aligned} \tag{43}$$

Here, the third equation follows from the relations

$$C_{23} \grave{R}^-_{12} = C_{12} \check{R}^-_{23}, \qquad C_{23} \hat{R}^-_{12} = C_{12} \grave{R}^-_{23} \tag{44}$$

which hold as the braiding induced by the action of the universal $R$-matrix is a natural isomorphism. In view of (8) and (10) Equation (43) implies

$$(q^{2(\mathrm{wt}(v_i)-\mathrm{wt}(v_j),\omega_s)} - q^{-(\alpha_s,\alpha_s)})\partial z_{ij} \in \mathcal{B}^+\Gamma'_\partial \qquad \text{for all } i,j \in I. \tag{45}$$

The relations (42) and (45) lead to

$$\partial z_{ij} \in \mathcal{B}^+\Gamma'_\partial \text{ if } j \neq N \text{ or } i \notin I_{(1)}. \tag{46}$$

This proves $\dim \Gamma'_\partial = \dim_{\mathbb{C}} \Gamma'_\partial / \mathcal{B}^+\Gamma'_\partial \leq \dim_{\mathbb{C}} V(\omega_s)_{(1)} = M$.

We now prove (iv). By the third relation of (30) the ideal $\mathcal{B}^+ \subset \mathcal{B}$ is generated by $\{z_{ij} \,|\, i \neq N \text{ or } j \neq N\}$. Equation (41) and (10) imply that $\partial z_{ij} z_{kl}$ can be written as a linear combination of elements $z_{mn}\partial z_{pt}$ where $\mathrm{wt}(v_k) \succ \mathrm{wt}(v_m)$ and $\mathrm{wt}(v_l) \succ \mathrm{wt}(v_n)$. This proves $\Gamma_\partial \mathcal{B}^+ \subset \mathcal{B}^+\Gamma_\partial$. The converse inclusion follows similarly from $z\partial z = q^{-(\alpha_s,\alpha_s)} \grave{R}^-_{23} \hat{R}^-_{12} \check{R}_{34} \acute{R}_{23} \partial z z$.

To prove (v) let $T$ denote the quantum tangent space of $\Gamma_\partial$. Recall from [HK03a, Lemma 6] and Remark 2.4(i) that the pairing

$$\Gamma_\partial / \mathcal{B}^+\Gamma_\partial \times T \to \mathbb{C}, \quad (\mathrm{d}b, f) \mapsto f(b)$$

is nondegenerate. Moreover, by [HK03b, Theorem 7.2] there exist precisely two non-isomorphic covariant FODC of dimension $M$ over $\mathcal{B}$. The corresponding quantum tangent spaces are $T_\partial = \mathrm{Lin}_{\mathbb{C}}\{F_\beta \,|\, \beta \in \overline{R^+_S}\}$ and $T_{\overline{\partial}} = \mathrm{Lin}_{\mathbb{C}}\{E_\beta \,|\, \beta \in \overline{R^+_S}\}$. As $T_{\overline{\partial}}$ vanishes on all $z_{iN}$, $i \in I_{(1)}$, relation (46) implies $T \neq T_{\overline{\partial}}$. □

### 3.2.5 The FODC $\Gamma_{\overline{\partial}}$

There exists a second covariant FODC $\Gamma_{\overline{\partial}}$ over $\mathcal{B}$ of dimension $\dim \mathfrak{g}/\mathfrak{p}_S$. This calculus can be obtained from a covariant FODC over $S_q[G/P_S^{\mathrm{op}}]$ in the same way as $\Gamma_\partial$ has been obtained from $\Gamma_+$. In analogy to $\Gamma_+$ a left $S_q[G/P_S^{\mathrm{op}}]$-module $\Gamma_-$ can be defined by generators $\mathrm{d}v_i$, $i \in I$, and relations

$$\sum_{i,j\in I} \left[\check{P}\check{Q}\right]_{ij}^{kl} v_i \mathrm{d}v_j = 0 \qquad \forall k,l \in I, \tag{47}$$

where as before $\check{P} = (\check{R} - q^{(\omega_s,\omega_s)}\mathrm{Id})$ and $\check{Q} := (\check{R} + q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)}\mathrm{Id})$. The left module $\Gamma_-$ can be endowed with a $S_q[G/P_S^{\mathrm{op}}]$-bimodule structure by

$$\mathrm{d}v_i v_j = q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)} \sum_{k,l\in I} \check{R}^{-}{}^{ij}_{kl} v_k \mathrm{d}v_l.$$

Defining the differential $\mathrm{d} : S_q[G/P_S^{\mathrm{op}}] \to \Gamma_-$ by $\mathrm{d}(v_i) = \mathrm{d}v_i$ and the Leibniz rule one obtains the desired covariant FODC over $S_q[G/P_S^{\mathrm{op}}]$. To construct a covariant FODC $(\Gamma_{-,\mathbb{C}}, \overline{\partial})$ over $S_q[G/P_S]_\mathbb{C}$ consider the left $S_q[G/P_S]_\mathbb{C}$-module $\Gamma_{-,\mathbb{C}} := S_q[G/P_S]_\mathbb{C} \otimes_{S_q[G/P_S^{\mathrm{op}}]} \Gamma_- \cong S_q[G/P_S] \otimes_\mathbb{C} \Gamma_-$. Then $\Gamma_{-,\mathbb{C}}$ can be endowed with a right $S_q[G/P_S]_\mathbb{C}$ module structure by

$$\mathrm{d}v_i f_j = q^{(\omega_s,\omega_s)} \sum_{k,l\in I} \grave{R}^{-}{}^{ij}_{kl} f_k \mathrm{d}v_l. \tag{48}$$

The differential $\overline{\partial} : S_q[G/P_S]_\mathbb{C} \to \Gamma_{-,\mathbb{C}}$ is defined by

$$\overline{\partial}(f_i) = 0, \quad \partial(v_i) = \mathrm{d}v_i.$$

and Leibniz rule. There exists a pairing

$$\begin{aligned} &\langle\, ,\, \rangle : \Gamma_{-,\mathbb{C}} \otimes V(\omega_s)^*_{(1)} \to \mathbb{C}, \\ &\langle w\mathrm{d}v, f\rangle := \varepsilon(w) f(v), \quad \text{ for } w \in S_q[G/P_S]_\mathbb{C},\ v \in S_q[G/P_S^{\mathrm{op}}]. \end{aligned} \tag{49}$$

In analogy to (36) one has

$$\overline{\partial}c\, f_k = f_k\overline{\partial}c, \quad \overline{\partial}c\, v_k = q^{-(\alpha_s,\alpha_s)} v_k \overline{\partial}c, \quad \mathrm{d}v_k\, c = c\mathrm{d}v_k + (q^{-(\alpha_s,\alpha_s)} - 1) v_k \overline{\partial}c. \tag{50}$$

Let $\Lambda \subset \Gamma_{-,\mathbb{C}}$ denote the subbimodule generated by $\overline{\partial}c$, $(c-1)\Gamma_{-,\mathbb{C}}$, and $\Gamma_{-,\mathbb{C}}(c-1)$. Then $\Gamma_{-,\mathbb{C}}/\Lambda$ is a covariant FODC over $S_q[G/P_S]_\mathbb{C}^{c=1}$ which as a left module is generated by $\mathrm{d}v_i$, $i \in I$, and relations (47) and $\overline{\partial}c = 0$. Again the pairing

$$\langle\, ,\, \rangle : \Gamma_{-,\mathbb{C}}/\Lambda \otimes V(\omega_s)^*_{(1)} \to \mathbb{C} \tag{51}$$

induced by (49) is well defined. Let $\Gamma_{\overline{\partial}} \subset \Gamma_{-,\mathbb{C}}/\Lambda$ denote the FODC over $\mathcal{B}$ induced by $\Gamma_{-,\mathbb{C}}/\Lambda$.

**Proposition 3.4.** *(i) As a left $\mathcal{B}$-module $\Gamma_{\overline{\partial}}$ is generated by the differentials $\overline{\partial} z_{ij}$, $i,j \in I$, and relations*

$$\check{P}_{34}\check{Q}_{34}\acute{R}_{23} z \overline{\partial} z = 0, \tag{52}$$

$$\hat{P}_{12}\acute{R}_{23} z \overline{\partial} z = 0, \tag{53}$$

$$\sum_{i,j\in I} C_{ij} \overline{\partial} z_{ij} = 0. \tag{54}$$

*(ii) The right $\mathcal{B}$-module structure of $\Gamma_{\overline{\partial}}$ is given by*

$$\overline{\partial} z z = q^{-(\alpha_s,\alpha_s)} \grave{R}^{-}_{23}\hat{R}_{12}\check{R}^{-}_{34}\acute{R}_{23} z \overline{\partial} z. \tag{55}$$

*(iii)* $\dim \Gamma_{\overline{\partial}} = M$.

*(iv)* $\Gamma_{\overline{\partial}}\mathcal{B}^+ = \mathcal{B}^+\Gamma_{\overline{\partial}}$.

*(v) The quantum tangent space of $\Gamma_{\overline{\partial}}$ is* $\mathrm{Lin}_{\mathbb{C}}\{E_\beta \,|\, \beta \in \overline{R_S^+}\}$.

*Proof.* The proof is performed in analogy to the proof of Proposition 3.3. The following remarks may be helpful. Let $\Gamma'_{\overline{\partial}}$ denote the left $\mathcal{B}$-module generated by elements $\overline{\partial} z_{ij}$, $i,j \in I$, and relations (52)-(54). Then relation (53) implies $\overline{\partial} z_{ij} = \sum_{k\in I} z_{ik}\overline{\partial} z_{kj}$ and therefore

$$\overline{\partial} z_{kl} \in \mathcal{B}^+\Gamma'_{\overline{\partial}} \quad \text{for all } k,l \in I \text{ such that } k \neq N. \tag{56}$$

Similarly to (43) one calculates

$$q^{(\omega_s,\omega_s)} C_{34}\grave{R}^{-}_{23}\hat{R}_{12}\check{R}^{-}_{34}\acute{R}_{23} z \overline{\partial} z = q^{(\alpha_s,\alpha_s)} \overline{\partial} z$$

which in view of (10) implies

$$(q^{2(\mathrm{wt}(v_i)-\mathrm{wt}(v_j),\omega_s)} - q^{(\alpha_s,\alpha_s)})\overline{\partial} z_{ij} \in \mathcal{B}^+\Gamma'_{\overline{\partial}}.$$

□

### 3.2.6 The FODC $\Gamma_{\mathrm{d}}$

To obtain a $q$-deformed analogue of classical Kähler differentials over $\mathbb{C}[G/L_S]$ we consider the sum

$$\Gamma_{\mathrm{d}} := \Gamma_\partial + \Gamma_{\overline{\partial}}. \tag{57}$$

**Corollary 3.5.** *(i) $\Gamma_{\mathrm{d}} = \Gamma_\partial \oplus \Gamma_{\overline{\partial}}$, in particular as a left $\mathcal{B}$-module $\Gamma_{\mathrm{d}}$ is generated by the elements $\partial z_{ij}, \overline{\partial} z_{ij}$, $i,j \in I$.*

*(ii)* $\dim \Gamma_{\rm d} = 2M$.

*(iii)* $\Gamma_{\rm d}\mathcal{B}^+ = \mathcal{B}^+\Gamma_{\rm d}$.

*Proof.* Recall that a sum of covariant FODC over $\mathcal{B}$ is direct if and only if the sum of their quantum tangent spaces is direct in $\mathcal{B}^\circ$. Now all statements of the corollary follow from Proposition 3.3 and 3.4. $\square$

## 3.3 Higher Order Differential Calculus

The aim of this subsection is to determine the dimensions of the homogeneous components of the universal DC $\Gamma^\wedge_{\partial,{\rm u}}$, $\Gamma^\wedge_{\overline{\partial},{\rm u}}$, and $\Gamma^\wedge_{{\rm d},{\rm u}}$ corresponding to the covariant FODC $\Gamma_\partial$, $\Gamma_{\overline{\partial}}$, and $\Gamma_{\rm d}$, respectively. As in the previous subsection we first focus on $\Gamma_\partial$.

### 3.3.1 The Differential Calculus $\Gamma^\wedge_{\partial,{\rm u}}$

Recall from Proposition 3.3(iv) that $\mathcal{B}^+\Gamma_\partial = \Gamma_\partial\mathcal{B}^+$ and hence $\Gamma^\wedge_{\partial,{\rm u}}/\mathcal{B}^+\Gamma^\wedge_{\partial,{\rm u}}$ is an algebra generated by $V_\partial := \Gamma_\partial/\mathcal{B}^+\Gamma_\partial$. For $i \in I_{(1)}$ let $x_i \in V_\partial$ denote the equivalence class of $\partial z_{iN} \in \Gamma_\partial$. Note that $V_\partial$ is an irreducible $K$-module isomorphic to $V(\omega_s)^*_{(1)}$ with one-dimensional weight spaces. Therefore each irreducible $K$-module in $V_\partial \otimes V_\partial$ occurs with multiplicity $\leq 1$. Hence the following notion makes sense. An irreducible $K$-submodule of $V_\partial \otimes V_\partial$ is called *(anti)symmetric* if the weight vectors of the corresponding classical $U(\mathfrak{l}_S)$-module are (anti)symmetric. Let $V_\partial \otimes V_\partial = S_\partial \oplus A_\partial$ denote the decomposition into the symmetric and antisymmetric subspaces.

In order to describe the algebra $\Gamma^\wedge_{\partial,{\rm u}}/\mathcal{B}^+\Gamma^\wedge_{\partial,{\rm u}}$ in terms of generators and relations it is useful to consider the $-\mathbb{N}_0$-filtration $\mathcal{H}$ on the vector space $V(\omega_s)^*_{(1)} \otimes V(\omega_s)^*_{(1)}$ defined by

$$\mathcal{H}_n(V(\omega_s)^*_{(1)} \otimes V(\omega_s)^*_{(1)}) = \mathrm{Lin}_{\mathbb{C}}\{E_\beta f_N \otimes E_\gamma f_N \mid \max(\mathrm{ht}(\beta), \mathrm{ht}(\gamma)) \geq -n\}$$

where $\mathrm{ht}(\sum_{i=1}^r n_i \alpha_i) = \sum_{i=1}^r n_i$. Moreover, we introduce the following notation: for any $\beta \in \overline{R^+_S}$ set $x_\beta := x_i$ where $i \in I_{(1)}$ and $\mathrm{wt}(f_i) = \mathrm{wt}(E_\beta f_N)$.

Consider the totally ordered abelian semigroup

$$\mathcal{N} = \{(k, n_1, \ldots, n_k) \mid k \in \mathbb{N}_0, n_i \in -\mathbb{N}, n_i \leq n_j \,\forall\, i < j\}$$

with the lexicographic order. The sum of two elements of $\mathcal{N}$ is defined by

$$(k, n_1, \ldots, n_k) + (l, m_1, \ldots, m_l) = (k + l, r_1, \ldots, r_{k+l})$$

where $r_1, \dots, r_{k+l}$ are the numbers $n_1, \dots, n_k, m_1, \dots, m_l$ in increasing order. The filtration $\mathcal{H}$ induces an $\mathcal{N}$-filtration on the algebra $\Gamma^{\wedge}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge}_{\partial,\mathrm{u}}$ defined by

$$\deg(x_\gamma) = (1, -\mathrm{ht}(\gamma)). \tag{58}$$

This filtration will also be denoted by $\mathcal{H}$.

**Proposition 3.6.** *(i) The algebra $\Gamma^{\wedge}_{\partial,\mathrm{u}}$ is generated by the elements $z_{ij}, \partial z_{ij}$, $i, j \in I$, and relations (30), (38)–(41), and*

$$\hat{Q}_{12}\acute{R}_{23}\partial z \wedge \partial z = 0, \quad \check{P}_{34}\acute{R}_{23}\partial z \wedge \partial z = 0. \tag{59}$$

*(ii) The algebra $\Gamma^{\wedge}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge}_{\partial,\mathrm{u}}$ is isomorphic to $V_\partial^{\otimes}/(S_\partial)$.*

*(iii) In the associated graded algebra $\mathrm{Gr}_{\mathcal{H}}\Gamma^{\wedge}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge}_{\partial,\mathrm{u}}$ the following relations hold:*

$$x_\beta \wedge x_\gamma + q^{(\beta,\gamma)} x_\gamma \wedge x_\beta = 0 \quad \textit{for all } \beta, \gamma \in \overline{R_S^+} \textit{ s. t. } \mathrm{ht}(\gamma) \le \mathrm{ht}(\beta).$$

*(iv) The set $\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \,|\, i_1 < i_2 < \cdots < i_k\}$ is a basis of $\Gamma^{\wedge k}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\partial,\mathrm{u}}$. In particular $\dim \Gamma^{\wedge k}_{\partial,\mathrm{u}} = \binom{M}{k}$.*

*Proof.* (i) Recall that by construction $\Gamma^{\wedge}_{\partial,\mathrm{u}}$ is the quotient of the tensor algebra $\Gamma^{\otimes}_\partial$ by the ideal generated by

$$\left\{ \sum_i \partial a_i \otimes \partial b_i \,\middle|\, \sum_i a_i \partial b_i = 0 \right\}.$$

By Proposition 3.3(i),(ii) this ideal is generated by

$$\{\hat{P}_{12}\hat{Q}_{12}\acute{R}_{23}\partial z \otimes \partial z, \check{P}_{34}\acute{R}_{23}\partial z \otimes \partial z, \partial z \otimes \partial z + q^{(\alpha_s,\alpha_s)}\grave{R}^-_{23}\hat{R}_{12}\check{R}^-_{34}\acute{R}_{23}\partial z \otimes \partial z\}$$

and therefore coincides with the ideal generated by

$$\{\hat{Q}_{12}\acute{R}_{23}\partial z \otimes \partial z, \check{P}_{34}\acute{R}_{23}\partial z \otimes \partial z\}.$$

(ii) We first prove that $\dim \Gamma^{\wedge 2}_{\partial,\mathrm{u}} = M(M-1)/2$. Let $T^{\varepsilon}_\Omega \subset U_q(\mathfrak{g})$ denote the right coideal generated by $\{K_\beta F_\beta \,|\, \beta \in \overline{R_S^+}\}$. Let $\Omega$ denote the left covariant FODC over $\mathcal{A}$ with quantum tangent space $T_\Omega = (T^{\varepsilon}_\Omega)^+$. By Proposition 3.3(v) the space $T_\Omega|_{\mathcal{B}} = T_\partial$ is the quantum tangent space of $\Gamma_\partial$. By Proposition 2.6 one has $\Omega|_{\mathcal{B}} = \Gamma_\partial$ and therefore Corollary 2.10 can be applied.

By Corollary 2.9 and Proposition 3.3(iv) one obtains $\dim_{\mathbb{C}} T_0 = M(\dim_{\mathbb{C}} T_\Omega - M)$. On the other hand consider the linear map

$$\mathrm{m} : T_\Omega \otimes T_\partial \to \overline{U}_-/T_\partial, \qquad s \otimes t \mapsto st$$

where $\overline{U}_- = U_q(\mathfrak{g})/U_q(\mathfrak{g})(K^+ + U_q(\mathfrak{b})^+)$. If $\beta_1, \dots, \beta_M$ denote the elements of $\overline{R_S^+}$ then by [HK03b, Prop. 5.2] the map m satisfies $\mathrm{Im}(\mathrm{m}) = \mathrm{Lin}_{\mathbb{C}}\{F_{\beta_i} F_{\beta_j} \,|\, i \leq j\}$ and hence $\dim \mathrm{Im}(\mathrm{m}) = M(M+1)/2$. By Corollary 2.10 this implies

$$\dim \Gamma^{\wedge 2}_{\partial,\mathrm{u}} = M(M-1)/2. \tag{60}$$

By the first equation of (59) and (10) the generators $x_i$ of $\Gamma^\wedge_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^\wedge_{\partial,\mathrm{u}}$ satisfy the relation

$$\sum_{k,l \in I_{(1)}} \hat{Q}^{ij}_{kl} x_k \wedge x_l = 0 \qquad \text{for all } i, j \in I.$$

For $\mathcal{Q} := \mathrm{Lin}_{\mathbb{C}}\{\sum_{k,l \in I_{(1)}} \hat{Q}^{ij}_{kl} x_k \otimes x_l \,|\, i, j \in I\} \subset V_\partial \otimes V_\partial$ by (60) one has $\dim_{\mathbb{C}} \mathcal{Q} \leq M(M+1)/2 = \dim_{\mathbb{C}} S_\partial$. Moreover, both $S_\partial$ and $\mathcal{Q}$ are $K$-submodules of $V_\partial \otimes V_\partial$. Therefore it suffices to show that the dimension of any weight space of $S_\partial$ does not exceed the dimension of the corresponding weight space of $\mathcal{Q}$.

For any element $\sum_{i,j \in I} a_{ij} f_i \otimes f_j \in V(2\omega_s)^* \subset V(\omega_s)^* \otimes V(\omega_s)^*$ where $a_{ij} \in \mathbb{C}$ one has

$$\sum_{i,j,k,l \in I} a_{ij} \hat{Q}^{ij}_{kl} f_k \otimes f_l = q^{(\omega_s,\omega_s)}(1 + q^{-(\alpha_s,\alpha_s)}) \sum_{i,j \in I} a_{ij} f_i \otimes f_j$$

and therefore $\sum_{i,j \in I_{(1)}} a_{ij} x_i \otimes x_j \in \mathcal{Q}$. For $\beta, \gamma \in \overline{R_S^+}$ one calculates

$$\begin{aligned}
E_\gamma E_\beta (f_N \otimes f_N) =& E_\gamma (q^{-(\alpha_s,\alpha_s)/2} E_\beta f_N \otimes f_N + f_N \otimes E_\beta f_N) \\
=& q^{-(\alpha_s,\alpha_s)} E_\gamma E_\beta f_N \otimes f_N + q^{-(\alpha_s,\alpha_s)/2} E_\beta f_N \otimes E_\gamma f_N \\
& + q^{(\beta,\gamma)-(\alpha_s,\alpha_s)/2} E_\gamma f_N \otimes E_\beta f_N + f_N \otimes E_\gamma E_\beta f_N \\
& + \sum_{i,j=1}^{n} a_{ij} E_{\beta_i} f_N \otimes E_{\beta_j} f_N
\end{aligned} \tag{61}$$

where in the last term $\beta_i, \beta_j \in \overline{R_S^+}$ such that $\max(\mathrm{ht}(\beta_i), \mathrm{ht}(\beta_j)) > \mathrm{ht}(\beta)$ and the complex numbers $a_{ij}$ depend on $\beta$ and $\gamma$. Then (61) implies that for every $\beta, \gamma \in \overline{R_S^+}$ with $\mathrm{ht}(\gamma) \leq \mathrm{ht}(\beta)$ there exists $v_{\beta,\gamma} \in \mathcal{H}_n(V(\omega_s)^*_{(1)} \otimes V(\omega_s)^*_{(1)})$, $n < -\mathrm{ht}(\beta)$, such that

$$x_\beta \otimes x_\gamma + q^{(\beta,\gamma)} x_\gamma \otimes x_\beta + v_{\beta,\gamma} \in \mathcal{Q}. \tag{62}$$

This implies that the dimension of any weight space of $S_\partial$ does not exceed the dimension of the corresponding weight space of $\mathcal{Q}$.

(iii) follows immediately from (62).

(iv) By (ii) the assertion holds for $k \le 2$. Moreover, (iii) implies that the set $\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \,|\, i_1 < i_2 < \cdots < i_k\}$ generates the vector spaces $\Gamma^{\wedge k}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\partial,\mathrm{u}}$. By the diamond lemma it suffices to prove the claim for $k = 3$. To this end define $V_{\overline{\partial}} := \Gamma_{\overline{\partial}}/\mathcal{B}^+\Gamma_{\overline{\partial}} \cong V(\omega_s)_{(1)}$ and let $V_{\overline{\partial}} \otimes V_{\overline{\partial}} = S_{\overline{\partial}} \oplus A_{\overline{\partial}}$ denote the decomposition into the symmetric and antisymmetric subspaces. By [HK03b, Cor. 6.7] the graded vector spaces $V_{\overline{\partial}}^{\otimes}/(A_{\overline{\partial}})$ and $\mathbb{C}[x_1, \ldots, x_M]$ are isomorphic. Thus one has

$$\dim(A_{\overline{\partial}} \otimes V_{\overline{\partial}} + V_{\overline{\partial}} \otimes A_{\overline{\partial}}) = M^3 - \binom{M+2}{3}.$$

The canonical pairing between $V(\omega_s)^*_{(1)}$ and $V(\omega_s)_{(1)}$ induces a nondegenerate pairing of $K$-modules

$$V_\partial^{\otimes 3} \otimes V_{\overline{\partial}}^{\otimes 3} \to \mathbb{C}, \quad (x'\otimes x''\otimes x''') \otimes (y'''\otimes y''\otimes y') \mapsto x'(y')x''(y'')x'''(y''').$$

With respect to this pairing the equation

$$S_\partial \otimes V_\partial \cap V_\partial \otimes S_\partial = (A_{\overline{\partial}} \otimes V_{\overline{\partial}} + V_{\overline{\partial}} \otimes A_{\overline{\partial}})^\perp$$

holds. Therefore

$$\begin{aligned}\dim_{\mathbb{C}}(S_\partial \otimes V_\partial + V_\partial \otimes S_\partial) &= 2M \dim_{\mathbb{C}} S_\partial - \dim(S_\partial \otimes V_\partial \cap V_\partial \otimes S_\partial)\\ &= M^2(M+1) - \binom{M+2}{3} = \binom{M}{3}\end{aligned}$$

which by (ii) implies the claim for $k = 3$. □

### 3.3.2 The Differential Calculus $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$

The situation for $\Gamma_{\overline{\partial}}$ is completely analogous. By Proposition 3.4(v) one has $\mathcal{B}^+\Gamma_{\overline{\partial}} = \Gamma_{\overline{\partial}}\mathcal{B}^+$ and hence $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$ is an algebra generated by $V_{\overline{\partial}} := \Gamma_{\overline{\partial}}/\mathcal{B}^+\Gamma_{\overline{\partial}}$. For $i \in I_{(1)}$ let $y_i \in V_{\overline{\partial}}$ denote the equivalence class of $\overline{\partial} z_{Ni} \in \Gamma_{\overline{\partial}}$. Moreover, we use the following notation: for any $\beta \in \overline{R_S^+}$ set $y_\beta := y_i$ where $i \in I_{(1)}$ and $\mathrm{wt}(f_i) = \mathrm{wt}(E_\beta f_N)$.

As in the proof of Proposition 3.6 let $V_{\overline{\partial}} \otimes V_{\overline{\partial}} = S_{\overline{\partial}} \oplus A_{\overline{\partial}}$ denote the decomposition into the symmetric and antisymmetric $K$-submodules. The algebra $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$ can be endowed with an $\mathcal{N}$-filtration defined by $\deg(y_\gamma) = (1, -\mathrm{ht}(\gamma))$. The proof of the following Proposition is a word by word translation of the proof of Proposition 3.6.

**Proposition 3.7.** *(i) The algebra* $\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}$ *is generated by the elements* $z_{ij}, \overline{\partial} z_{ij}$, $i,j \in I$, *and relations (30), (52)–(55) and*

$$\hat{P}_{12}\acute{R}_{23}\overline{\partial}z \wedge \overline{\partial}z = 0, \quad \check{Q}_{34}\acute{R}_{23}\overline{\partial}z \wedge \overline{\partial}z = 0. \tag{63}$$

*(ii) The algebra* $\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}$ *is isomorphic to* $V^{\otimes}_{\overline{\partial}}/(S_{\overline{\partial}})$.
*(iii) In the associated graded algebra* $\mathrm{Gr}_{\mathcal{H}}\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}$ *the following relations hold:*

$$y_\beta \wedge y_\gamma + q^{-(\beta,\gamma)} y_\gamma \wedge y_\beta = 0 \quad \textit{for all } \beta,\gamma \in \overline{R^+_S} \textit{ s. t. } \mathrm{ht}(\gamma) \le \mathrm{ht}(\beta).$$

*(iv) The set* $\{y_{i_1} \wedge y_{i_2} \wedge \cdots \wedge y_{i_k} \,|\, i_1 < i_2 < \cdots < i_k\}$ *is a basis of* $\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}}$. *In particular* $\dim \Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}} = \binom{M}{k}$.

### 3.3.3 Extending $\partial$ and $\overline{\partial}$ to $\Gamma^{\wedge}_{\mathrm{d},\mathrm{u}}$

Our next aim is to obtain results for $\Gamma^{\wedge}_{\mathrm{d},\mathrm{u}}$ analogous to Propositions 3.3 and 3.4. To this end we first show that the decomposition $\Gamma_{\mathrm{d}} = \Gamma_\partial \oplus \Gamma_{\overline{\partial}}$ induces differentials $\partial$ and $\overline{\partial}$ on $\Gamma^{\wedge}_{\mathrm{d},\mathrm{u}}$ such that $\mathrm{d} = \partial + \overline{\partial}$ also holds in higher degrees.

**Proposition 3.8.** *There exists a uniquely determined linear map* $\partial : \Gamma^{\wedge}_{\mathrm{d},\mathrm{u}} \to \Gamma^{\wedge}_{\mathrm{d},\mathrm{u}}$ *such that*

- *(i)* $\partial(\mathcal{B}) \subset \Gamma_\partial \subset \Gamma_{\mathrm{d}}$ *and* $\partial|_{\mathcal{B}}$ *coincides with the differential* $\partial$ *considered in Section 3.2.*
- *(ii)* $\partial(\mathrm{d}a) = -\mathrm{d}(\partial a)$ *for all* $a \in \mathcal{B}$.
- *(iii)* $(\Gamma^{\wedge}_{\mathrm{d},\mathrm{u}} = \bigoplus_{i\in\mathbb{N}_0} \Gamma^{\wedge i}_{\mathrm{d},\mathrm{u}}, \partial)$ *is a differential graded algebra.*

*The map* $\partial$ *satisfies* $\partial\mathrm{d} = -\mathrm{d}\partial$.

*Proof.* Uniqueness holds as $\mathcal{B}$ and $\mathrm{d}\mathcal{B}$ generate the algebra $\Gamma^{\wedge}_{\mathrm{d},\mathrm{u}}$. To prove existence the following auxiliary lemma is needed. Let $T_{\Omega,E}$ and $T_{\Omega,F}$ denote the intersection of $\ker\varepsilon$ with the right coideal of $U_q(\mathfrak{g})$ generated by $\{E_\beta \,|\, \beta \in \overline{R^+_S}\}$ and $\{K_\beta F_\beta \,|\, \beta \in \overline{R^+_S}\}$, respectively. By (4) the sum $T_\Omega = T_{\Omega,E} + T_{\Omega,F} \subset U_q(\mathfrak{g})$ is direct. Moreover, $T_\Omega$ is the quantum tangent space of a left covariant FODC $\Omega$ over $\mathcal{A}$ such that $\Omega|_{\mathcal{B}} = \Gamma_{\mathrm{d}}$. Let $\pi_{\Omega,E}, \pi_{\Omega,F} : T_\Omega = T_{\Omega,E} \oplus T_{\Omega,F} \to T_\Omega$ and $\pi_E, \pi_F : T = T_\partial \oplus T_{\overline{\partial}} \to T$ denote the canonical projections onto $T_{\Omega,E}$, $T_{\Omega,F}$, $T_\partial$, and $T_{\overline{\partial}}$, respectively. Recall the pairings (22) and (24).

**Lemma 3.9.** *The pairings*

$$\langle\cdot,\cdot\rangle : \Gamma_{\mathrm{d}} \times T \to \mathbb{C}, \qquad \langle\!\langle\cdot,\cdot\rangle\!\rangle : \mathcal{A} \otimes_{\mathcal{B}} \Gamma_{\mathrm{d}} \otimes_{\mathcal{B}} \Gamma_{\mathrm{d}} \times T_\Omega \otimes T \to \mathbb{C}$$

*satisfy the relations*

$$\langle \partial a, t\rangle = \langle \mathrm{d}a, \pi_F t\rangle, \qquad \langle \overline{\partial} a, t\rangle = \langle \mathrm{d}a, \pi_E t\rangle, \tag{64}$$
$$\langle\!\langle \partial a \otimes \rho, s \otimes t\rangle\!\rangle = \langle\!\langle \mathrm{d}a \otimes \rho, \pi_{\Omega,F} s \otimes t\rangle\!\rangle, \tag{65}$$
$$\langle\!\langle \overline{\partial} a \otimes \rho, s \otimes t\rangle\!\rangle = \langle\!\langle \mathrm{d}a \otimes \rho, \pi_{\Omega,E} s \otimes t\rangle\!\rangle, \tag{66}$$
$$\langle\!\langle \rho \otimes \partial a, s \otimes t\rangle\!\rangle = \langle\!\langle \rho \otimes \mathrm{d}a, s \otimes \pi_F t\rangle\!\rangle, \tag{67}$$
$$\langle\!\langle \rho \otimes \overline{\partial} a, s \otimes t\rangle\!\rangle = \langle\!\langle \rho \otimes \mathrm{d}a, s \otimes \pi_E t\rangle\!\rangle \tag{68}$$

*for all* $a \in \mathcal{B}$*,* $\rho \in \Gamma_{\mathrm{d}}$*,* $s \in T_\Omega$*,* $t \in T$*.*

*Proof of Lemma 3.9.* To prove (64) recall that by (46) and Proposition 3.3(i)

$$\partial a \in \mathrm{Lin}_{\mathbb{C}}\{\partial z_{iN} \,|\, i \in I_{(1)}\} + \mathcal{B}^+\Gamma_\partial \quad \text{ for all } a \in \mathcal{B}.$$

Moreover, (56) implies $\overline{\partial} z_{iN} \in \mathcal{B}^+\Gamma_{\mathrm{d}}$ if $i \in I_{(1)}$. Therefore using Corollary 3.5(i) and $\partial z_{iN} = \mathrm{d}z_{iN} - \overline{\partial} z_{iN}$ one obtains

$$\partial a \in \mathrm{Lin}_{\mathbb{C}}\{\mathrm{d} z_{iN} \,|\, i \in I_{(1)}\} + \mathcal{B}^+\Gamma_{\mathrm{d}} \quad \text{ for all } a \in \mathcal{B}.$$

Since $(\pi_E t)(z_{iN}) = 0$ for all $i \in I_{(1)}$ this implies $\langle \partial a, \pi_E t\rangle = 0$ and hence

$$\langle \partial a, t\rangle = \langle \partial a, \pi_F t\rangle \quad \text{ for all } a \in \mathcal{B},\ t \in T.$$

Analogously one obtains

$$\langle \overline{\partial} a, t\rangle = \langle \overline{\partial} a, \pi_E t\rangle \quad \text{ for all } a \in \mathcal{B},\ t \in T$$

which yields (64). The remaining formulae follow from (64) and the definition (24) using

$$\left.\begin{aligned} \pi_{\Omega,F} s^+_{(0)} \otimes s_{(1)} &= (\pi_{\Omega,F} s)^+_{(0)} \otimes (\pi_{\Omega,F} s)_{(1)}, \\ \pi_{\Omega,E} s^+_{(0)} \otimes s_{(1)} &= (\pi_{\Omega,E} s)^+_{(0)} \otimes (\pi_{\Omega,E} s)_{(1)} \end{aligned}\right\} \quad \text{ for all } s \in T_\Omega.$$

□

We continue with the proof of Proposition 3.8. The first step to prove existence of the map $\partial : \Gamma^\wedge_{\mathrm{d,u}} \to \Gamma^\wedge_{\mathrm{d,u}}$ is to show that

$$\partial : \Gamma^{\wedge 1}_{\mathrm{d,u}} \to \Gamma^{\wedge 2}_{\mathrm{d,u}}, \quad \partial(a\mathrm{d}b) := \partial a \wedge \mathrm{d}b - a\, \mathrm{d}\partial b \tag{69}$$

is well defined. Assume that $\sum_i a_i \mathrm{d}b_i = 0$. Then $\sum_i a_i \partial b_i = 0$ and hence

$$\begin{aligned} \sum_i (\partial a_i \wedge \mathrm{d}b_i - a_i \mathrm{d}\partial b_i) &= \sum_i (\partial a_i \wedge \mathrm{d}b_i + \mathrm{d}a_i \wedge \partial b_i) \\ &= \sum_i (2\partial a_i \wedge \partial b_i + \partial a_i \wedge \overline{\partial} b_i + \overline{\partial} a_i \wedge \partial b_i). \end{aligned}$$

Observe that $\{\sum_i(2\partial a_i \wedge \partial b_i + \partial a_i \wedge \overline{\partial} b_i + \overline{\partial} a_i \wedge \partial b_i) \mid \sum_i a_i \mathrm{d} b_i = 0\}$ is a left $\mathcal{A}$-subcomodule of $\Gamma^{\wedge 2}_{\mathrm{d,u}}$. Thus by Corollary 2.10 it suffices to show that

$$\sum_{i,j} \langle\!\langle (2\partial a_i \wedge \partial b_i + \partial a_i \wedge \overline{\partial} b_i + \overline{\partial} a_i \wedge \partial b_i), s_j \otimes t_j \rangle\!\rangle = 0 \tag{70}$$

whenever $\sum_i a_i \mathrm{d} b_i = 0$ and $\sum_j s_j t_j \in T$. By Lemma 3.9 the left hand side of (70) is equal to

$$\sum_{i,j} \langle\!\langle \mathrm{d} a_i \wedge \mathrm{d} b_i, s_j \otimes t_j + \pi_{\Omega,F} s_j \otimes \pi_F t_j - \pi_{\Omega,E} s_j \otimes \pi_E t_j \rangle\!\rangle.$$

As $\sum_j \pi_{\Omega,F} s_j \pi_F t_j$, $\sum_j \pi_{\Omega,E} s_j \pi_E t_j \in T$ whenever $\sum_j s_j t_j \in T$, the relation $\sum_i \mathrm{d} a_i \wedge \mathrm{d} b_i = 0$ implies (70) and therefore (69) is well defined.

To verify that $\partial$ is well defined on $\Gamma^{\wedge}_{\mathrm{d,u}}$ it suffices to check that

$$\sum_i \partial \mathrm{d} a_i \wedge \mathrm{d} b_i - \mathrm{d} a_i \wedge \partial \mathrm{d} b_i = 0$$

whenever $\sum_i a_i \mathrm{d} b_i = 0$, $a_i, b_i \in \mathcal{B}$. This follows from

$$\partial \mathrm{d} a \wedge \mathrm{d} b - \mathrm{d} a \wedge \partial \mathrm{d} b = -\mathrm{d}(\partial a \wedge \mathrm{d} b - a \mathrm{d} \partial b) = -\mathrm{d}(\partial(a \mathrm{d} b))$$

for all $a, b \in \mathcal{B}$.

The property $\partial \mathrm{d} \rho = -\mathrm{d} \partial \rho$ for all $\rho \in \Gamma^{\wedge k}_{\mathrm{d,u}}$ is proved by induction over $k$.

Next we show that $(\partial \circ \partial)|_{\mathcal{B}} = 0$. To this end we calculate the adjoint operators of d and $\partial$ with respect to the pairing (27). Assume that $\sum_j s_j t_j \in T$, $s_j \in T_\Omega$, $t_j \in T$, then

$$\begin{aligned}
\left\langle\!\!\left\langle \mathrm{d}(a \mathrm{d} b), \sum_j s_j \otimes t_j \right\rangle\!\!\right\rangle &= \left\langle\!\!\left\langle \mathrm{d} a \otimes \mathrm{d} b, \sum_j s_j \otimes t_j \right\rangle\!\!\right\rangle \\
&\overset{(25)}{=} \sum_j s_j(a^+ b_{(-1)}) t_j(b_{(0)}) \\
&= \sum_j s_j(a b_{(-1)}) t_j(b_{(0)}) - \varepsilon(a) s_j t_j(b)
\end{aligned}$$

and hence

$$\left\langle\!\!\left\langle \mathrm{d}\rho, \sum_j s_j \otimes t_j \right\rangle\!\!\right\rangle = \sum_j s_j(\rho_{(-1)}) \langle \rho_{(0)}, t_j \rangle - \left\langle \rho, \sum_j s_j t_j \right\rangle \tag{71}$$

for all $\rho \in \Gamma^{\wedge 1}_{\mathrm{d,u}}$. Similarly, for all $a, b \in \mathcal{B}$, using Lemma 3.9, one obtains

$$\begin{aligned}\left\langle\!\!\left\langle \partial(a\mathrm{d}b), \sum_j s_j \otimes t_j \right\rangle\!\!\right\rangle =& \left\langle\!\!\left\langle \partial a \wedge \mathrm{d}b + \mathrm{d}a \wedge \partial b - \mathrm{d}(a\partial b), \sum_j s_j \otimes t_j \right\rangle\!\!\right\rangle \\ \overset{(71)}{=}& \left\langle\!\!\left\langle \mathrm{d}a \otimes \mathrm{d}b, \sum_j (\pi_{\Omega,F} s_j \otimes t_j + s_j \otimes \pi_F t_j) \right\rangle\!\!\right\rangle \\ &- \sum_j s_j(a_{(-1)}b_{(-1)})\langle a_{(0)}\partial b_{(0)}, t_j\rangle + \left\langle a\partial b, \sum_j s_j t_j \right\rangle \\ \overset{(71)}{=}& \sum_j \pi_{\Omega,F} s_j(a_{(-1)}b_{(-1)})\langle a_{(0)}\mathrm{d}b_{(0)}, t_j\rangle \\ &- \left\langle a\mathrm{d}b, \sum_j [(\pi_{\Omega,F} s_j)t_j + s_j(\pi_F t_j) - \pi_F(s_j t_j)] \right\rangle.\end{aligned}$$

This leads to

$$\begin{aligned}\left\langle\!\!\left\langle \partial^2 a, \sum_j s_j \otimes t_j \right\rangle\!\!\right\rangle =& \sum_j \pi_{\Omega,F} s_j(a_{(-1)})\langle \partial a_{(0)}, t_j\rangle \\ &- \left\langle \partial a, \sum_j [s_j t_j - \pi_{\Omega,E} s_j \pi_E t_j + \pi_{\Omega,F} s_J \pi_F t_j - \pi_F(s_j t_j)] \right\rangle \\ =& \sum_j \pi_{\Omega,F} s_j(a_{(-1)})\pi_F t_j(a_{(0)}) - \left\langle \mathrm{d}a, \sum_j \pi_{\Omega,F} s_j \pi_F t_j \right\rangle = 0.\end{aligned}$$

It remains to prove $\partial^2\rho = 0$ for all $\rho \in \Gamma^{\wedge k}_{\mathrm{d,u}}$, $k \geq 1$, which is obtained by induction over $k$. Assume $\partial^2\omega = 0$ then using $\partial\mathrm{d} = -\mathrm{d}\partial$ one gets

$$\partial^2(\mathrm{d}a \wedge \omega) = \partial(-\mathrm{d}\partial a \wedge \omega - \mathrm{d}a \wedge \partial\omega) = \mathrm{d}(\partial^2 a) \wedge \omega + \mathrm{d}a \wedge \partial^2\omega = 0.$$

□

**Remark 3.10.** For the map $\overline{\partial} : \Gamma^{\wedge}_{\mathrm{d,u}} \to \Gamma^{\wedge}_{\mathrm{d,u}}$ defined by $\overline{\partial} := \mathrm{d} - \partial$ one immediately obtains properties analogous to Proposition 3.8. Moreover, one verifies that $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

### 3.3.4 The Differential Calculus $\Gamma^{\wedge}_{\mathrm{d,u}}$

Now we are prepared to write the algebra $\Gamma^{\wedge}_{\mathrm{d,u}}$ in terms of generators and relations and to calculate $\dim \Gamma^{\wedge k}_{\mathrm{d,u}}$ for all $k$. By Propositions 3.3(v) and 3.4(v) one has $\mathcal{B}^+\Gamma_{\mathrm{d}} = \Gamma_{\mathrm{d}}\mathcal{B}^+$ and hence $\Gamma^{\wedge}_{\mathrm{d,u}}/\mathcal{B}^+\Gamma^{\wedge}_{\mathrm{d,u}}$ is an algebra generated by $\Gamma_{\mathrm{d}}/\mathcal{B}^+\Gamma_{\mathrm{d}} = \Gamma_\partial/\mathcal{B}^+\Gamma_\partial \oplus \Gamma_{\overline{\partial}}/\mathcal{B}^+\Gamma_{\overline{\partial}}$. Recall that for $\beta \in \overline{R^+_S}$ we write $x_\beta$ and $y_\beta$ to denote the equivalence class of $\partial z_{iN} \in \Gamma_\partial$ and $\overline{\partial} z_{Ni} \in \Gamma_{\overline{\partial}}$ for

suitable $i \in I_{(1)}$, respectively. The algebra $\Gamma^\wedge_{\mathrm{d,u}}/\mathcal{B}^+\Gamma^\wedge_{\mathrm{d,u}}$ can be endowed with an $\mathcal{N}$-filtration defined by $\deg(x_\gamma) = (1, -\mathrm{ht}(\gamma)) = \deg(y_\gamma)$. As before this $\mathcal{N}$-filtration will be denoted by $\mathcal{H}$.

**Proposition 3.11.** *(i) The algebra $\Gamma^\wedge_{\mathrm{d,u}}$ is generated by the elements $z_{ij}$, $\partial z_{ij}$, $\overline{\partial} z_{ij}$, $i,j \in I$, and relations (30), (38) – (41), (52) – (55), (59), (63), and*

$$\overline{\partial} z \wedge \partial z = -q^{-(\alpha_s,\alpha_s)} T^-_{1234} \partial z \wedge \overline{\partial} z + q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)} z C_{12} T^-_{1234} \partial z \wedge \overline{\partial} z \quad (72)$$

*where $T^-_{1234} = \grave{R}^-_{23} \hat{R}^-_{12} \check{R}_{34} \acute{R}_{23}$.*

*(ii) The algebra $\Gamma^\wedge_{\mathrm{d,u}}/\mathcal{B}^+\Gamma^\wedge_{\mathrm{d,u}}$ is isomorphic to $(V_\partial \oplus V_{\overline{\partial}})^\otimes/(S_\partial + S_{\overline{\partial}} + J)$ where $J \subset (V_\partial \otimes V_{\overline{\partial}}) \oplus (V_{\overline{\partial}} \otimes V_\partial)$ is the subspace spanned by all expressions of the form*

$$y_i \otimes x_j + q^{(\omega_s,\omega_s)-(\alpha_s,\alpha_s)} \sum_{k,l \in I_{(1)}} \grave{R}^{-ij}_{\ \ kl} x_k \otimes y_l, \quad i,j \in I_{(1)}. \quad (73)$$

*(iii) In the associated graded algebra $\mathrm{Gr}_\mathcal{H} \Gamma^\wedge_{\mathrm{d,u}}/\mathcal{B}^+\Gamma^\wedge_{\mathrm{d,u}}$ the following relations hold:*

$$y_\beta \wedge y_\gamma + q^{-(\beta,\gamma)} y_\gamma \wedge y_\beta = 0,$$
$$x_\beta \wedge x_\gamma + q^{(\beta,\gamma)} x_\gamma \wedge x_\beta = 0$$

*for all $\beta, \gamma \in \overline{R^+_S}$ such that $\mathrm{ht}(\gamma) \le \mathrm{ht}(\beta)$, and*

$$y_\beta \wedge x_\gamma + q^{-(\beta,\gamma)} x_\gamma \wedge y_\beta = 0$$

*for all $\beta, \gamma \in \overline{R^+_S}$.*

*(iv) For all $k \in \mathbb{N}_0$ the canonical map*

$$\bigoplus_{i+j=k} \Gamma^{\wedge i}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge i}_{\partial,\mathrm{u}} \otimes \Gamma^{\wedge j}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge j}_{\overline{\partial},\mathrm{u}} \to \Gamma^{\wedge k}_{\mathrm{d,u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\mathrm{d,u}} \quad (74)$$

*is an isomorphism. In particular $\dim \Gamma^{\wedge k}_{\mathrm{d,u}} = \binom{2M}{k}$.*

*Proof.* (i) By Corollary 3.5 the algebra $\Gamma^\wedge_{\mathrm{d,u}}$ is generated by the elements $z_{ij}$, $\partial z_{ij}$, $\overline{\partial} z_{ij}$, $i,j \in I$. Moreover, Propositions 3.3 and 3.4 imply that the relations (38) – (41) and (52) – (55) hold. Applying $\partial$ and $\overline{\partial}$ one obtains (59) and (63). In the following we verify (72).

Using (44) and applying $C_{23}\check{R}^-_{34}$ to (39) and $C_{23}\hat{R}^-_{12}$ to (53) one obtains for $D := C\acute{R}$ the relations

$$D_{23} z\, \partial z = 0, \qquad\qquad D_{23} z\, \overline{\partial} z = \overline{\partial} z.$$

Leibniz rule for $\overline{\partial}$ yields $D_{23}\overline{\partial} z\, z = 0$ and $D_{23}\overline{\partial} z \wedge \partial z = D_{23} z\, \partial\overline{\partial} z$. Thus one gets

$$\partial\overline{\partial} z = D_{23}\partial(z\overline{\partial} z) = D_{23}(\partial z \wedge \overline{\partial} z + \overline{\partial} z \wedge \partial z).$$

With the abbreviation $T_{1234} = \grave{R}^-_{23}\hat{R}_{12}\check{R}^-_{34}\acute{R}_{23}$ using (41) and (55) this leads to

$$\begin{aligned}
\partial\overline{\partial} z\, z &= D_{23}(\partial z \wedge \overline{\partial} z + \overline{\partial} z \wedge \partial z) z = D_{23} T_{3456} T_{1234}(z\partial z \wedge \overline{\partial} z + z\overline{\partial} z \wedge \partial z)\\
&= D_{23}\grave{R}^-_{45}\hat{R}_{34}\grave{R}^-_{23}\hat{R}_{12}\check{R}^-_{56}\acute{R}_{45}\check{R}^-_{34}\acute{R}_{23}(z\partial z \wedge \overline{\partial} z + z\overline{\partial} z \wedge \partial z)\\
&= \grave{R}^-_{23} D_{34}\hat{R}_{12}\check{R}^-_{56}\acute{R}_{45}\check{R}^-_{34}\acute{R}_{23}(z\partial z \wedge \overline{\partial} z + z\overline{\partial} z \wedge \partial z)\\
&= \grave{R}^-_{23}\hat{R}_{12}\check{R}^-_{34} D_{45}\acute{R}_{23}(z\partial z \wedge \overline{\partial} z + z\overline{\partial} z \wedge \partial z)\\
&= T_{1234} z\partial\overline{\partial} z
\end{aligned} \tag{75}$$

where the relations

$$D_{12}\hat{R}_{23}\grave{R}^-_{12} = D_{23}, \qquad D_{12}\acute{R}_{23}\check{R}^-_{12} = D_{23}$$

have been used. Now $\overline{\partial}$ is applied to (41) which leads to

$$\overline{\partial}\partial z\, z - \partial z \wedge \overline{\partial} z = q^{(\alpha_s,\alpha_s)} T_{1234}\overline{\partial} z \wedge \partial z + q^{(\alpha_s,\alpha_s)} T_{1234} z\overline{\partial}\partial z.$$

In view of (75) multiplication by $T^-_{1234}$ yields

$$(1 - q^{(\alpha_s,\alpha_s)}) z\overline{\partial}\partial z = T^-_{1234}\partial z \wedge \overline{\partial} z + q^{(\alpha_s,\alpha_s)}\overline{\partial} z \wedge \partial z. \tag{76}$$

Application of $C_{12}$ leads to

$$q^{-(\omega_s,\omega_s)}(1 - q^{(\alpha_s,\alpha_s)})\overline{\partial}\partial z = C_{12} T^-_{1234}\partial z \wedge \overline{\partial} z.$$

Inserting this formula in (76) one finally gets the desired Equation (72).

Now Propositions 3.6(iv), 3.7(iv) and (72) imply

$$\dim \Gamma_{\mathrm{d}}^{\otimes 2}/\Lambda \le \binom{2M}{2}$$

where $\Lambda \subset \Gamma_{\mathrm{d}}^{\otimes 2}$ denotes the $\mathcal{B}$-bimodule corresponding to the relations (59), (63), and (72). Let $\Omega$ denote the left covariant FODC over $\mathcal{A}$ defined in the

proof of Proposition 3.8. Recall that $\Omega|_{\mathcal{B}} = \Gamma_{\mathrm{d}}$ and $T_\Omega|_{\mathcal{B}} = T$ and therefore Corollary 2.10 can be applied. By Corollaries 2.9 and 3.5(iii) one obtains $\dim_{\mathbb{C}} T_0 = 2M(\dim_{\mathbb{C}} T_\Omega - 2M)$. On the other hand let $\beta_1, \dots, \beta_M$ denote the elements of $\overline{R_S^+}$. By [HK03b, Prop. 5.2] the linear map

$$\mathrm{m} : T_\Omega \otimes T \to \overline{U}/T, \qquad s \otimes t \mapsto st.$$

satisfies $\mathrm{Im}(\mathrm{m}) = \mathrm{Lin}_{\mathbb{C}}\{F_{\beta_i}F_{\beta_j}, E_\beta F_\gamma, E_{\beta_i}E_{\beta_j} \,|\, i \leq j,\, \beta, \gamma \in \overline{R_S^+}\}$ and therefore $\dim \mathrm{Im}(\mathrm{m}) = 2M(2M+1)/2$. By Corollary 2.10 one obtains

$$\dim \Gamma_{\mathrm{d,u}}^{\wedge 2} = \binom{2M}{2}. \tag{77}$$

This implies $\Gamma_{\mathrm{d}}^{\otimes 2}/\Lambda = \Gamma_{\mathrm{d,u}}^{\wedge 2}$ and completes the proof of (i) as $\Gamma_{\mathrm{d,u}}^{\wedge}$ is a quadratic algebra.

(ii) The algebra $\Gamma_{\mathrm{d,u}}^{\wedge}/\mathcal{B}^+\Gamma_{\mathrm{d,u}}^{\wedge}$ is generated by the elements $x_i, y_i$, $i = 1, \dots, M$, and the relations induced by (59), (63), and (72). It has already been stated in Proposition 3.6 and Proposition 3.7 that the relations induced by (59) and (63) are obtained by setting $S_\partial \subset V_\partial \otimes V_\partial$ and $S_{\overline{\partial}} \subset V_{\overline{\partial}} \otimes V_{\overline{\partial}}$ equal to zero. On the other hand (10) and (72) imply that (73) vanishes in $\Gamma_{\mathrm{d,u}}^{\wedge}/\mathcal{B}^+\Gamma_{\mathrm{d,u}}^{\wedge}$. By (77) there can be no more quadratic relations.

(iii) The first two relations of (iii) have already been stated in Propositions 3.6 and 3.7. In view of (10) and the definition of the filtration $\mathcal{H}$ the last relation follows from (73).

(iv) By Equation (59), (63), and (72) the vector space $\Gamma_{\mathrm{d,u}}^{\wedge}/\mathcal{B}^+\Gamma_{\mathrm{d,u}}^{\wedge}$ is a quotient of the tensor product $(\Gamma_{\partial,\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\partial,\mathrm{u}}^{\wedge}) \otimes (\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge})$. Moreover, the map (74) is an isomorphism if and only if there exists a product $\wedge$ on $(\Gamma_{\partial,\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\partial,\mathrm{u}}^{\wedge}) \otimes (\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge})$ which extends the algebra structures of $\Gamma_{\partial,\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\partial,\mathrm{u}}^{\wedge}$ and $\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge}$ and satisfies (73). Existence of the product $\wedge$ follows from Lemma 2.2 and the naturality of the braiding (8) of $U_q([\mathfrak{l}_S, \mathfrak{l}_S])$. $\square$

**Remark 3.12.** (i) For $\mathfrak{g} = \mathfrak{o}_n$, $s = 1$, and $\mathfrak{g} = \mathfrak{sl}_n$ the algebras $\Gamma_{\partial,\mathrm{u}}^{\wedge}/\mathcal{B}^+\Gamma_{\partial,\mathrm{u}}^{\wedge}$ are well known examples of quantized exterior algebras [CP94, Def. 7.4.4], [FRT89].

(ii) Propositions 3.11(iv), 3.6(iv), and 3.7(iv) imply that $\Gamma_{\mathrm{d,u}}^{\wedge 2M}/\mathcal{B}^+\Gamma_{\mathrm{d,u}}^{\wedge 2M}$ is a one dimensional trivial $K$-module. Thus $\Gamma_{\mathrm{d,u}}^{\wedge 2M}$ is a free left $\mathcal{B}$-module generated by one left coinvariant element. In contrast the covariant differential calculi $\Gamma_{\partial,\mathrm{u}}^{\wedge}$ and $\Gamma_{\overline{\partial},\mathrm{u}}^{\wedge}$ do not admit a volume form.

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